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EFFECT OF A HIGH INITIAL STRESS ON THE PROPAGATION OF LOVE WAVES IN AN ISOTROPIC ELASTIC INCOMPRESSIBLE MEDIUM*

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The problem of Love waves in an incompressible elastic medium stressed initially by a uniform homogeneous deformation is investigated. The effect of the initial stress in a neo-Hookean material on Love waves is analysed.

1. INTRODUCTION

The material of the earth being in a state of initial stress, the study of surface wave propagation in a pre-stressed medium is important. Effect of initial stress on the wave propagation has been analysed by Hencky (1932), Biot (1940), Hayes and Rivlin (1961a), Flavin (1962), Green (1963), and others. Hayes and Rivlin (1961b) have also analysed the problem of surface waves in the general context. Recently, Nowinski (1977) has investigated the problem of Love wave propagation in an incompressible isotropic layered half space of neo-Hookean materials subjected to uniaxial initial compression parallel to the plane free surface. In this paper we extend the analysis of Nowinski to the case when the layered half space is initially homogeneously deformed in all its principal directions. The finite elasticity method as enunciated by Green, is used to find the frequency equation, which is then analysed. The results obtained are compared with the classical results and the results obtained by Nowinski (1977).

2. FORMULATION OF THE PROBLEM

Let us consider an isotropic homogeneous elastic half space $\bar{B}_0$, $z \geqslant 0$ covered by an isotropic homogeneous elastic layer B_0, $-h \leqslant z < 0$ whose mechanical properties are different from those of $\bar{B}_0$, and assume that the layer and the half space are in welded contact. Let the system $B_0 + \bar{B}_0$ undergo a uniform homogeneous deformation along x, y, z axes which are along the principal directions, such that the system goes to the system $B + \bar{B}$. We take θ_i with the body $B + \bar{B}$ and identify them as Cartesian coordinates x, y, z in $B + \bar{B}$ and the coordinates x_0, y_0, z_0 in $B_0 + \bar{B}_0$. Thus (x_0, y_0, z_0) goes to (x, y, z) by the homogeneous deformation :

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$$x = \lambda_1 x_0,\ y = \lambda_2 y_0,\ z = \lambda_3 z_0,\ \theta_1 = x,\ \theta_2 = y,\ \theta_3 = z. \qquad ...(2.1)$$

The fundamental metric tensor in $B + \bar{B}$ is $G_{ij} = G^{ij} = \delta_{ij}$, $i, j = 1, 2, 3$. The determinant of G_{ij} is 1.

The fundamental metric tensor in $B_0 + \bar{B}_0$ is given by

$$g_{ij} = \begin{bmatrix} 1/\lambda_1^2 & 0 & 0 \\ 0 & 1/\lambda_2^2 & 0 \\ 0 & 0 & 1/\lambda_3^2 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix},$$

$$g = | g_{ij} | = \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2}. \qquad ...(2.3)$$

The invariants of the above finite strain are

$$I_1 = g^{ij} G_{ij} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\ I_3 = G/g = \lambda_1^2 \lambda_2^2 \lambda_3^2,$$

$$I_2 = g_{ij} G^{ij} I_3 = \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2. \qquad ...(2.4)$$

We take the material homogeneous, isotropic and incompressible. The strain energy function W is a function of I_1, I_2 and the initial stresses are given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + p G^{ij} \qquad ...(2.5)$$

where $B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs}$, $\Phi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_1}$, $\Psi = \frac{2}{\sqrt{I_3}} \frac{\partial W}{\partial I_2}$ and p is an arbitrary constant which is to be determined by boundary conditions. The incompressibility condition is

$$I_3 = 1 = \lambda_1^2 \lambda_2^2 \lambda_3^2. \qquad ...(2.6)$$

Thus
$$\tau^{11} = \Phi \lambda_1^2 + \Psi \lambda_1^2 (\lambda_2^2 + \lambda_3^2) + p,\quad \tau^{22} = \Phi \lambda_2^2 + \Psi \lambda_2^2 (\lambda_1^2 + \lambda_3^2) + p,$$

$$\tau^{33} = \Phi \lambda_3^2 + \Psi \lambda_3^2 (\lambda_1^2 + \lambda_2^2) + p,$$

$$\tau^{12} = \tau^{21} = \tau^{32} = \tau^{23} = \tau^{13} = \tau^{31} = 0.$$

Assuming that the surface $z = -h$ is free from stresses, we have the boundary condition as $\tau^{33} = 0$ at $z = -h$ and since τ^{33} is constant, it is zero throughout the medium. Hence $p = -\Phi \lambda_3^2 - \Psi \lambda_3^2 (\lambda_1^2 + \lambda_2^2)$ and putting this value of p in other stress components, we finally get the initial stresses as

$$\left.\begin{aligned} &\tau^{11} = (\Phi + \lambda_2^2\Psi)(\lambda_1^2 - \lambda_3^2),\ \tau^{22} = (\Phi + \lambda_1^2\Psi)(\lambda_2^2 - \lambda_3^2), \\ &\tau^{33} = \tau^{12} = \tau^{23} = \tau^{31} = 0 \quad \text{in } -h \leqslant z < 0, \\ \text{and}\quad & \\ &\bar{\tau}^{11} = (\bar{\Phi} + \lambda_2^2\bar{\Psi})(\lambda_1^2 - \lambda_3^2),\ \bar{\tau}^{22} = (\bar{\Phi} + \lambda_1^2\bar{\Psi})(\lambda_2^2 - \lambda_3^2), \\ &\text{in } z \geqslant 0. \end{aligned}\right\} \qquad \dots(2.7)$$

Since by construction, the displacements of the layer and half space are equal and $\tau^{31} = \tau^{32} = \tau^{33} = 0$, the continuity at $z = 0$ is preserved. Let us now consider an infinitesimal deformation superimposed on the system $B + \bar{B}$, having the displacement components given by

$$(0, \epsilon v, 0), \quad \text{and} \quad (0, \epsilon\bar{v}, 0) \text{ in } B \text{ and } \bar{B} \text{ respectively.} \qquad \dots(2.8)$$

This deformation transforms $B + \bar{B}$ to $B' + \bar{B}'$ with stresses $\tau^{ij} + \epsilon\tau'^{ij}$, the metric tensor in $B' + \bar{B}'$ being $G'_{ij} + \epsilon G'_{ij}$ where ϵ is very small. Assuming that v, $\bar{v}$ an functions of x, t, it can be shown, following Green, that

$$\left.\begin{aligned} &G'_{ij} = \begin{bmatrix} 0 & \dfrac{\partial v}{\partial x} & 0 \\ \dfrac{\partial v}{\partial x} & 0 & \dfrac{\partial v}{\partial z} \\ 0 & \dfrac{\partial v}{\partial z} & 0 \end{bmatrix},\ G'^{ij} = -G^{ir}G^{js}G'_{rs} = -G'_{ij} \\ &G' = |G'_{ij}| = 0 \end{aligned}\right\} \qquad \dots(2.9)$$

and strain invariants are $I_1 + \epsilon I'_1$, $I_2 + \epsilon I'_2$, $I_3 + \epsilon I'_3$, where

$$I'_1 = g^{ij}G'_{ij} = 0,\ I'_3 = G'/g = 0,\ I'_2 = g_{ij}(G'^{ij}I_3 + G^{ij}I'_3) = 0 \qquad \dots(2.10)$$

Also following Green and Zerna (1954), we get superposed stresses as

$$\tau'^{11} = \tau'^{22} = \tau'^{33} = p',\ \tau'^{23} = c_{44}\frac{\partial v}{\partial z},\ \tau'^{12} = c_{66}\frac{\partial v}{\partial z},\ \tau'^{13} = 0$$

$$\text{in } -h \leqslant z < 0$$

$$\bar{\tau}'^{11} = \bar{\tau}'^{22} = \bar{\tau}'^{33} = \bar{p}',\ \bar{\tau}'^{23} = \bar{c}_{44}\frac{\partial \bar{v}}{\partial z},\ \bar{\tau}'^{12} = \bar{c}_{66}\frac{\partial \bar{v}}{\partial x},\ \bar{\tau}'^{13} = 0 \text{ in } z \geqslant 0 \qquad \dots(2.11)$$

where

$$\left.\begin{aligned} c_{44} &= -\Psi\lambda_2^2\lambda_3^2 - p = \lambda_3^2(\Phi + \lambda_1^2\Psi) \quad \text{in } -h \leqslant z < 0 \\ c_{66} &= -\Psi\lambda_1^2\lambda_2^2 - p = \lambda_3^2(\Phi + \lambda_1^2\Psi) + \lambda_2^2\Psi(\lambda_3^2 - \lambda_1^2) \text{ in } -h \leqslant z < 0. \\ &\text{and} \\ \bar{c}_{44} &= \lambda_3^2(\bar{\Phi} + \lambda_1^2\bar{\Psi}),\ \bar{c}_{66} = \lambda_3^2(\bar{\Phi} + \lambda_1^2\bar{\Psi}) + \lambda_2^2\bar{\Psi}(\lambda_3^2 - \lambda_1^2) \quad \text{in } z \geqslant 0. \end{aligned}\right\} \quad \ldots(2.12)$$

The equations of linear momentum are

$$\frac{\partial}{\partial\theta^i}\left(\tau'^{ij} + \tau^{ir}\frac{\partial u_j}{\partial\theta^r} + \tau^{rj}\frac{\partial u_i}{\partial\theta^r}\right) = \rho\,\frac{\partial^2 u_j}{\partial t^2}$$

where $\theta^i \equiv (x, y, z),\ u_i = (0, v, 0)$.

These yield the following equations:

$$\frac{\partial\tau'^{11}}{\partial x} = \frac{\partial\tau'^{33}}{\partial z} = 0,$$

$$\frac{\partial\tau'^{12}}{\partial x} + \frac{\partial\tau'^{32}}{\partial z} + \frac{\partial}{\partial x}\left(\tau^{11}\frac{\partial v}{\partial x}\right) = \rho\,\frac{\partial^2 v}{\partial t^2}, \quad \text{in } -h \leqslant z < 0,$$

$$\frac{\partial\bar{\tau}'^{11}}{\partial x} = \frac{\partial\bar{\tau}'^{33}}{\partial \bar{z}} = 0,$$

$$\frac{\partial\bar{\tau}'^{12}}{\partial x} + \frac{\partial\bar{\tau}'^{32}}{\partial z} + \frac{\partial}{\partial x}\left(\bar{\tau}^{11}\frac{\partial \bar{v}}{\partial x}\right) = \bar{\rho}\,\frac{\partial^2 v}{\partial t^2}, \quad \text{in } z \geqslant 0. \quad \ldots(2.13)$$

The 1st, 2nd, 4th and 5th of eqn. (2.13) show that p' and $\bar{p}'$ are functions of t only. As the boundary is free, $\tau'^{33}(x, -h, t) = 0$. Also from continuity of stress across $z = 0$, $\tau'^{33}(x, 0, t) = \bar{\tau}'^{33}(x, 0, t)$. We get $p' = \bar{p}' = 0$ identically. Therefore $\tau'^{11} = \tau'^{22} = \tau'^{33} = \tau'^{13} = 0$, and $\bar{\tau}'^{11} = \bar{\tau}'^{22} = \bar{\tau}'^{33} = \bar{\tau}'^{13} = 0$ identically, and we are to solve the 3rd and last of eqns. (2.13) which become, on substituting from (2.11),

$$\left.\begin{aligned} &c_x^2\,\frac{\partial^2 v}{\partial x^2} + c_z^2\,\frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial t^2} \quad \text{in } -h \leqslant z < 0 \\ &\text{and} \\ &\bar{c}_x^2\,\frac{\partial^2 \bar{v}}{\partial x^2} + \bar{c}_z^2\,\frac{\partial^2 \bar{v}}{\partial z^2} = \frac{\partial^2 \bar{v}}{\partial t^2} \quad \text{in } z \geqslant 0 \end{aligned}\right\} \quad \ldots(2.15)$$

where

$$\left.\begin{aligned} c_x^2 &= \frac{\lambda_1^2}{\rho}(\Phi + \lambda_3^2\Psi),\quad c_z^2 = \frac{\lambda_3^2}{\rho}(\Phi + \lambda_1^2\Psi) \\ \bar{c}_x^2 &= \frac{\lambda_1^2}{\bar{\rho}}(\bar{\Phi} + \lambda_3^2\bar{\Psi}),\quad \bar{c}_z^2 = \frac{\lambda_3^2}{\bar{\rho}}(\bar{\Phi} + \lambda_1^2\bar{\Psi}). \end{aligned}\right\} \quad \ldots(2.16)$$

Assuming that a Love-type wave is propagating along x-direction, we take the solution of (2.15) as

$$\left.\begin{aligned} v &= V(z)\, e^{ik(x-ct)} \quad \text{in } -h \leqslant z < 0 \\ \bar{v} &= \bar{V}(z)\, e^{ik(x-ct)} \quad \text{in } z \geqslant 0. \end{aligned}\right\} \qquad \ldots(2.17)$$

We get the equation for V and $\bar{V}$ as

$$\left.\begin{aligned} &\frac{d^2V}{dz^2} + k^2\sigma^2 V = 0 \quad \text{in } -h \leqslant z < 0, \\ &\qquad \text{where } \sigma^2 = (c^2 - c_x^2)/c_z^2 \\ &\frac{d^2\bar{V}}{dz^2} - k^2\bar{\sigma}^2\bar{V} = 0 \quad \text{in } z > 0, \\ &\qquad \text{where } \bar{\sigma}^2 = (\bar{c}_x^2 - c^2)/\bar{c}_z^2 \,. \end{aligned}\right\} \qquad \ldots(2.18)$$

The solution of (2.18) is

$$\left.\begin{aligned} V &= A \cos k\sigma z + B \sin k\sigma z \quad \text{in } -h \leqslant z < 0 \\ \bar{V} &= \bar{A} e^{-k\bar{\sigma} z} \quad \text{in } z > 0. \end{aligned}\right\} \qquad \ldots(2.19)$$

(2.19) must satisfy the boundary condition

$$\tau'^{23} = 0 \quad \text{at } z = -h \qquad \ldots(2.20a)$$

and the condition of continuity at

$$\left.\begin{aligned} v &= \bar{v} \quad \text{at } z = 0 \\ \tau'^{23} &= \bar{\tau}'^{23} \quad \text{at } z = 0. \end{aligned}\right\} \qquad \ldots(2.20b)$$

Substituting (2.19) in (2.20a) and (2.20b), we get

$$\tan\left[kh\left(\frac{c^2}{c_z^2} - \frac{c_x^2}{c_z^2}\right)^{1/2}\right] = \frac{\bar{c}_{44}}{c_{44}}\left[\frac{(\bar{c}_x^2 - c^2)\, c_z^2}{(c^2 - c_x^2)\,\bar{c}_z^2}\right]^{1/2}. \qquad \ldots(2.21)$$

Equation (2.21) is the frequency equation of Love wave in the pre-stressed medium concerned which shows that the velocity c of Love wave is contained in the interval $c_x < c < \bar{c}_x$ which also implies that for Love wave propagation, wave velocity $\bar{c}_x$ in substratum is greater than the wave velocity c_x in the layer. If the superficial layer is absent ($h = 0$), then by the condition that $\bar{\tau}'^{23} = 0$ at $z = 0$, we conclude that Love-type of wave is still possible in the neighbourhood of the surface under very restrictive condition that $\bar{c}_{44} = 0$ i.e. $\bar{\Phi} = -\lambda_1^2\Psi$ as was observed by Hayes and

Rivlin (1961b). Also if the initial stress or the initial deformation is absent i.e. $\lambda_1 = \lambda_2 = \lambda_3 = 1$, eqn. (2.21) coincides with the famous classical Love frequency equation.

3. Love Waves in Neo-Hookean Medium

In neo-Hookean type material, the strain energy function W is given by

$$W = c_1(I_1 - 3).$$

$$\Phi = 2c_1,\ \Psi = 0,\ \bar{\Phi} = 2\bar{c}_1,\ \bar{\Psi} = 0. \qquad \ldots(3.1)$$

In the limiting case of an infinitesimal deformation the constitutive equation reduces to Hooke's law and thus we get

$$\beta^2 = \Phi/\rho = \mu/\rho,\ \bar{\beta}^2 = \bar{\Phi}/\bar{\rho} = \bar{\mu}/\bar{\rho}, \quad \text{where } \mu, \bar{\mu} \text{ are rigidity moduli.} \qquad \ldots(3.2)$$

Thus from (2.16),

$$c_x^2 = \lambda_1^2\beta^2,\ \bar{c}_x^2 = \lambda_1^2\bar{\beta}^2,\ c_z^2 = \lambda_3^2\beta^2 = \frac{\beta^2}{\lambda_1^2\lambda_2^2},\ \bar{c}_z^2 = \lambda_3^2\bar{\beta}^2 = \frac{\bar{\beta}^2}{\lambda_1^2\lambda_2^2} \qquad \ldots(3.3)$$

and from (2.12)

$$c_{44} = \lambda_3^2\Phi = \frac{\rho\beta^2}{\lambda_1^2\lambda_2^2},\ \bar{c}_{44} = \frac{\bar{\rho}\bar{\beta}^2}{\lambda_1^2\lambda_2^2}. \qquad \ldots(3.4)$$

Substituting (3.3) and (3.4) in (2.21), we get frequency equation in neo-Hookean material as

$$\tan\left[kh\lambda_1\lambda_2\{(c^2/\beta^2) - \lambda_1^2\}^{1/2}\right] = \frac{\bar{\mu}}{\mu}\left[\frac{\lambda_1^2 - (c^2/\bar{\beta}^2)}{(c^2/\beta^2) - \lambda_1^2}\right]^{1/2}. \qquad \ldots(3.5)$$

The classical result, with no initial deformation, is obtained from (3.5) by putting $\lambda_1 = \lambda_2 = 1$, namely,

$$\tan\left[kh((c^2/\beta^2) - 1)^{1/2}\right] = \frac{\bar{\mu}}{\mu}\left[\frac{1 - (c^2/\bar{\beta}^2)}{(c^2/\beta^2) - 1}\right]^{1/2} \qquad \ldots(3.6)$$

4. Particular Cases

We now consider three particular cases:

Case 1 — When $\lambda_2 = 1$, (3.5) reduces to

$$\tan\left[kh\lambda_1\left((c^2/\beta^2)-\lambda_1^2\right)^{1/2}\right]=\frac{\bar{\mu}}{\mu}\left[\frac{\lambda_1^2-(c^2/\bar{\beta}^2)}{(c^2/\beta^2)-\lambda_1^2}\right]^{1/2}. \qquad \ldots(4.1)$$

This result has been obtained by Nowinski (1977).

Case 2 — When $\lambda_1 = 1$, i.e. initially the medium is compressed uniaxially along a direction parallel to the free surface and perpendicular to the direction of propagation of wave, (3.5) reduces to

$$\tan\left[kh\lambda_2\left((c^2/\beta^2)-1\right)^{1/2}\right]=\frac{\bar{\mu}}{\mu}\left[\frac{1-(c^2/\bar{\beta}^2)}{(c^2/\beta^2)-1}\right]^{1/2}. \qquad \ldots(4.2)$$

This result when compared with classical result shows that, for a given value of $\bar{\mu}/\mu$, $\bar{\rho}/\rho$, the phase velocity c for a fixed wavenumber k in classical case is equal to the plase velocity in this case when the wavenumber is k/λ_2. Fig. 1 shows the graph of c/β for values of l/h, where l is the wavelength $(= 2\pi/k)$, with $\bar{\rho}/\rho = 1$, $\bar{\mu}/\mu = 1.25$ in our case and is compared with the classical results shown in dotted lines. Thus when $\lambda_2 > 1$, for given value of wavelength, the phase velocity in the present case is less than that of the classical case and when $\lambda_2 < 1$, it is otherwise.

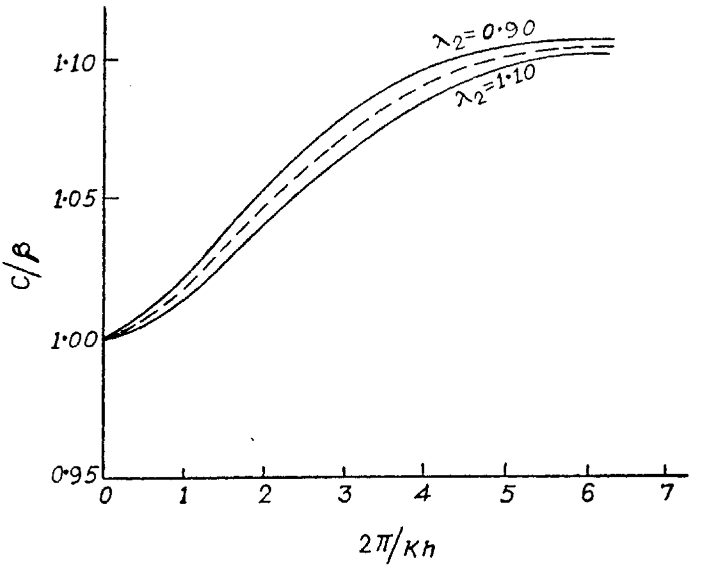

FIG. 1.

Case 3 — When $\lambda_1\lambda_2 = 1$, i.e. $\lambda_3 = 1$, (3.5) becomes

$$\tan\left[kh\left((c^2/\beta^2)-\lambda_1^2\right)^{1/2}\right]=\frac{\bar{\mu}}{\mu}\left[\frac{\lambda_1^2-(c^2/\bar{\beta}^2)}{(c^2/\beta^2)-\lambda_1^2}\right]^{1/2}. \qquad \ldots(4.3)$$

Equation (4.3) when compared with Nowinski's result (4.1), shows that, if $\bar{\mu}/\mu$, $\bar{\rho}/\rho$ are known, then for a fixed wavenumber k, the phase velocity in this case is equal to the value of phase velocity corresponding to $k\lambda_1$ in (4.1).

In Fig. 2, we have plotted the velocity against wavelength for $\lambda_1 = 0.95$, 0.90 and compared with Nowinski's results shown in dotted lines. We find that for a given value of wavelength, with $\bar{\rho}/\rho = 1$, $\bar{\mu}/\mu = 1.25$, the phase velocity c in the present case is less than that of Nowinski for $\lambda_1 < 1$.

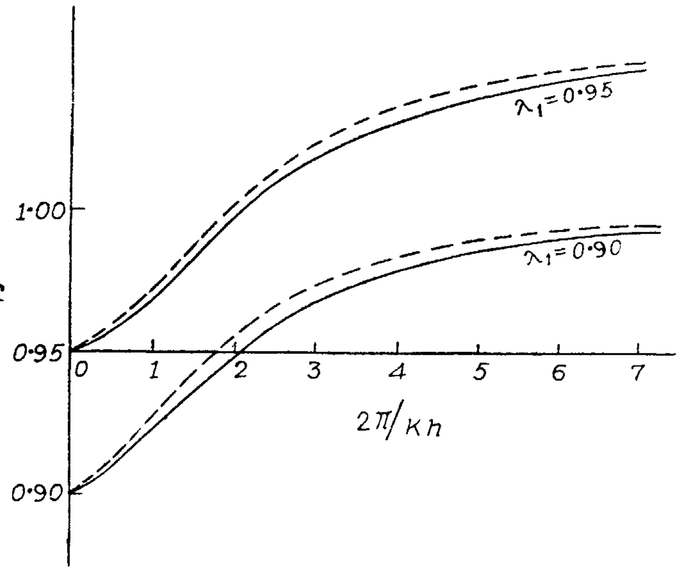

FIG. 2.

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