Indian J. pure appl. Math., **10**(9) : 1107-1115, September 1979

VISCOUS STRATIFIED FLOW IN AN OSCILLATING CYLINDER

K. B. NAIDU*

Department of Mathematics, Sri Venkateswara University, Tirupati 517502

(*Received* 13 *September* 1978)

An initial value problem is considered for the motion of a density- and viscous-stratified fluid bounded externally by an infinite vertical circular cylinder, when the cylinder acquires a velocity of oscillatory form parallel to its generators. At time $t = 0$, the cylindrical boundary begins to move with velocity $W \sin \omega t$ in the axial direction. We assume that the amplitude of oscillation is much less than the stratification height so that the parameter $\varepsilon = \beta W/\omega << 1$. In view of the geometrical configuration and boundary conditions of the problem we consider that the flow is uni-directional and seek solutions of the form $u = 0$, $w = w(r, t)$ whereas pressure, density and viscosity are dependent on z also. Expanding the flow variables w, P, ρ, μ in the exact non-linear equations, in ascending powers of ε we take the lowest order equations (which are linear) as the governing equations of our problem and solve by Laplace transform method.

The solution contains an oscillatory part which relates to steady oscillation with the same frequency as that of the boundary (but with a phase difference) and a transient part which decays to zero as time tends to infinity. When the frequency of oscillation of the cylinder is equal to the Brunt-Vaisala frequency, the flow is everywhere in phase with the oscillating boundary for all nonzero values of kinematic viscosity. Expressions for velocity are also obtained (i) when the radius of the cylinder is large and (ii) for small time (but not small r/a). The skin friction on the surface of the cylinder is evaluated when the diffusion length is much less than the radius of the cylinder; and it is found that the effects of surface curvature and perturbation in density are not significant in this case.

1. INTRODUCTION

Exact solution for the commencement of Hagen-Poiseuille flow of a homogeneous viscous fluid in a circular cylinder was obtained by Szymanski (1932). The solution shows that under the influence of difference of pressure at the ends of the cylinder, the central portion of the fluid has an approximately constant velocity across the section but increasing with time, whereas at the cylindrical boundary friction retards the motion of the fluid. As time passes, the velocity distribution tends to be parabolic. The periodic motion of a homogeneous viscous fluid in a long cylinder under

*Present address: Department of Mathematics, P.S.G. College of Arts & Science, Civil Aerodrome Post, Coimbatore 641014.

the influence of periodic pressure gradient was investigated experimentally by Tyler and Richardson (1929) and theoretically by Sexl (1930). In this case also the velocity distribution is of parabolic form. If the distance from the wall of the cylinder is much less than the radius of the cylinder, the boundary layer approximation can be applied to the equations [by neglecting $(1/r)\,\partial w/\partial r$] and the solution can be obtained in a simple form.

Impulsively started axial motion of a cylinder in a homogeneous viscous fluid has been considered by Wu and Yao-Tsu Wu (1967) [a generalized version of Rayleigh's (1911) problem] in which the flow is uni-directional and the velocity satisfies the two-dimensional diffusion equation with no convective effect, so that the vorticity generated at the boundary is diffused into the fluid by viscous action only.

Forced oscillations in an inviscid stratified fluid have been considered by many authors (Sarma and Krishna 1969; Hendershott 1969; Rao and Rao 1971; Sarma and Naidu 1972a, b; etc.). But very little work has been done on the oscillations in stratified viscous fluids.

In this paper, an initial value problem is considered for the motion of a density- and viscous-stratified fluid bounded externally by an infinite vertical circular cylinder.

2. Formulation of the Problem

In this paper we consider an initial value problem of the motion of a density- and viscous-stratified fluid, bounded externally by an infinite vertical circular cylinder, when the cylinder acquires a velocity of oscillatory form $W \sin \omega t$ parallel to its generators.

We refer the equations of fluid motion to fixed cylindrical coordinate system (r, θ, z) with z-axis (directed vertically upwards) coinciding with the axis of the cylinder. In the equilibrium state, the density and viscosity are taken as $\rho_0(z)$ and $\mu_0(z)$ (Dore 1969), where

$$\frac{\rho_0(z)}{\rho_0'} = \frac{\mu_0(z)}{\mu_0'} = \exp(-\beta z) \qquad \ldots(1)$$

(ρ_0' and μ_0' being the mean density and viscosity, and β the stratification parameter) so that the equilibrium pressure $\tilde{P}_0(z)$ satisfies

$$\frac{d\tilde{P}_0}{dz} = -g\rho_0 \qquad \ldots(2)$$

and the Brunt-Vaisala frequency

$$N = \left(- \frac{g}{\rho_0} \frac{d\rho_0}{dz} \right)^{1/2} = \sqrt{(\beta g)} = \left(- \frac{g}{\mu_0} \frac{d\mu_0}{dz} \right)^{1/2} \quad ...(3)$$

is constant.

We assume that the amplitude of oscillation is much less than the stratification height so that the parameter

$$\epsilon = \beta W/\omega \ll 1. \quad ...(4)$$

In view of the geometrical configuration and boundary conditions of the problem, we consider that the flow is uni-directional and seek a solution of the form

$$u = 0, \; w = w(r, t) \quad ...(5)$$

whereas pressure, density and viscosity are dependent on z also.

The exact non-linear equations governing the axisymmetric motion of a density- and viscous-stratified fluid are [using (2)]

$$(\rho_0 + \rho)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial r} + \frac{\partial}{\partial r}\left((\mu_0 + \mu)\, 2\frac{\partial u}{\partial r}\right)$$
$$+ \frac{\partial}{\partial z}\left((\mu_0 + \mu)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right) + \frac{2}{r}(\mu_0 + \mu)\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) \quad ...(6)$$

$$(\rho_0 + \rho)\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} - \rho g + \frac{\partial}{\partial z}\left((\mu_0 + \mu)\, 2\frac{\partial w}{\partial z}\right)$$
$$+ \frac{1}{r}\frac{\partial}{\partial r}\left((\mu_0 + \mu)\, r\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\right) \quad ...(7)$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + w\left(\frac{d\rho_0}{dz} + \frac{\partial \rho}{\partial z}\right) = 0 \quad ...(8)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad ...(9)$$

where u is the radial velocity, w the axial velocity, and P, ρ, μ are perturbation pressure, density and viscosity respectively.

With the assumptions (4) and (5), we set

$$u = 0 \quad ...(10)$$

$$w(r, t) = W(w_0(r, t) + \epsilon w_1(r, t) + ...) \quad ...(11)$$

$$P(r, z, t) = W\beta\mu_0' \exp(-\beta z)\,(P_0(r, t) + \epsilon P_1(r, t) + ...) \quad ...(12)$$

$$\rho(r, z, t) = \epsilon\rho_0' \exp(-\beta z)\,(\sigma_0(r, t) + \epsilon\sigma_1(r, t) + ...) \quad ...(13)$$

$$\mu(r, z, t) = \epsilon\mu_0' \exp(-\beta z)(\gamma_0(r, t) + \epsilon\gamma_1(r, t) + \ldots) \qquad \ldots(14)$$

where $\epsilon = W\beta/\omega$ ($\ll 1$). Substituting (11) - (14) in eqns. (6) - (9), equating the terms of the lowest order in ϵ and eliminating P_0 and σ_0 from the resulting equations, we obtain the governing equation for w_0

$$\nu \frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \left(\frac{\partial^2 w}{\partial t^2} + \nu\beta^2 \frac{\partial w}{\partial t} + N^2 w\right) = 0 \qquad \ldots(15)$$

where the subscript '0' is deleted and $\nu = \mu_0'/\rho_0'$. From the form of this equation, it is evident that the terms involving ν will be dominant when $t \to +\infty$ if the frequency of oscillation of the cylinder is very close to the Brunt-Vaisälä frequency. We have to solve eqn (15) with the initial and boundary conditions

$$w = \frac{dw}{dt} = 0 \quad \text{for} \quad 0 \leqslant r \leqslant a \quad \text{when} \quad t \leqslant 0 \qquad \ldots(16)$$

$$w = \sin \omega t \quad \text{at} \quad r = a \quad \text{when} \quad t > 0. \qquad \ldots(17)$$

3. Solution and Discussion

If we apply the Laplace transform defined by

$$\bar{w}(r, s) = \int_0^\infty \exp(-st)\, w(r, t)\, dt \qquad \ldots(18)$$

(real part of s being greater than 0)

to eqns. (15) - (17), $\bar{w}(r, s)$ satisfies the ordinary differential equation

$$\frac{d^2\bar{w}}{dr^2} + \frac{1}{r}\frac{d\bar{w}}{dr} - q^2\bar{w} = 0 \qquad \ldots(19)$$

and the condition

$$\bar{w} = \frac{\omega}{s^2 + \omega^2} \quad \text{at} \quad r = a \qquad \ldots(20)$$

where

$$q = ((s^2 + \nu\beta^2 s + N^2)/\nu s)^{1/2}. \qquad \ldots(21)$$

Of the two solutions $I_0(r \cdot q)$ and $K_0(r \cdot q)$ (modified Bessel functions of the first and second kind) of the eqn. (19), only I_0 is to be taken for our problem (since $I_0 \to 1$, $K_0 \to \infty$ as $r \to 0$; and $\bar{w}$ must be finite at $r = 0$). Therefore the solution of (19) with the boundary condition (20) is

$$\bar{w}(r, s) = \frac{\omega}{s^2 + \omega^2}\frac{I_0(rq)}{I_0(aq)}. \qquad \ldots(22)$$

Now we obtain $w(r, t)$ through the inversion formula

$$w(r, t) = \frac{\omega}{2\pi i} \int_{Br} \frac{\exp(st)\, I_0(rq)\, ds}{(s^2 + \omega^2)\, I_0(aq)} \qquad \text{...(23)}$$

where Br is the Bromwich contour from $b - i\infty$ to $b + i\infty$ in the complex s-plane, b being a real positive constant greater than the real part of all singularities of $\bar{w}(r, s)$. The series of $I_0(z)$ shows that it is a function of even powers of z, so that $I_0(rq)/I_0(aq)$ is a single-valued function of s. The zeros of $I_0(aq)$ are at

$$s_n = \tfrac{1}{2}\left(-\nu(\beta^2 + \alpha_n^2) \pm (\nu^2(\beta^2 + \alpha_n^2)^2 - 4N^2)^{1/2}\right), \quad (n = 1, 2, 3, \ldots) \qquad \text{...(24)}$$

where $\pm\alpha_n (n = 1, 2, 3, \ldots)$ are the roots (all real and simple) of $J_0(a\alpha) = 0$. Hence the poles of the integrand in (23) are at $s = \pm i\omega$ and at $s = s_n$ $(n = 1, 2, 3, \ldots)$ where s_n are given by (24). Thus, for a given n, if $\nu^2(\beta^2 + \alpha_n^2)^2 < 4N^2$ the corresponding poles are complex-conjugates situated on the circle $|s| = N$; and if $\nu^2(\beta^2 + \alpha_n^2)^2 > 4N^2$ the poles lie on the negative real axis and are inverse points with respect to the same circle. All these are simple poles, but a single double pole $s_j = -N$ occurs if $\nu^2(\beta^2 + \alpha_n^2)^2 = 4N^2$. The line integral in (23) can be written as

$$\int_{Br} \frac{\exp(st)\, I_0(rq)}{(s^2 + \omega^2)\, I_0(aq)}\, ds = 2\pi i \,[\text{sum of the residues at the poles}] - \int_{C} \frac{\exp(st)\, I_0(rq)}{(s^2 + \omega^2)\, I_0(aq)}\, ds \qquad \text{...(25)}$$

where C is the circular arc of large radius on the left of Br (Carslaw and Jaeger 1959). The approximations for $I_0(rq)$ and $I_0(aq)$ show that the integral round this arc C can be shown to tend to zero as the radius of the arc tends to infinity. Thus, when all the singularities of the integrand are simple poles, we obtain the velocity field.

$$w(r, t) = \frac{1}{2i}\left(\exp(i\omega t)\frac{I_0(r\sqrt{L})}{I_0(a\sqrt{L})} - \exp(-i\omega t)\frac{I_0(r\sqrt{M})}{I_0(a\sqrt{M})}\right) + \frac{2\nu\omega}{a}\sum_{n=1}^{\infty}\frac{\exp(s_n t)\, J_0(r\alpha_n)\, \alpha_n s_n^2}{(s_n^2 + \omega^2)(s_n^2 - N^2)\, J_1(a\alpha_n)} \qquad \text{...(26)}$$

where

$$\left.\begin{aligned} L &= [\nu\omega\beta^2 + i(\omega^2 - N^2)]/\nu\omega \\ M &= [\nu\omega\beta^2 - i(\omega^2 - N^2)]/\nu\omega \end{aligned}\right\} \quad \ldots(27)$$

and where the summation is over both the values of s_n given in (24). In the case of a double pole occurring at $s = s_j = -N$, the contribution is evaluated as

$$\frac{2\nu\omega}{a} \frac{\sqrt{K} J_0(r\sqrt{K}) \exp(-Nt) [(\omega^2 - N^2) - Nt(\omega^2 + N^2)]}{J_1(a\sqrt{K})(\omega^2 + N^2)^2} \quad \ldots(28)$$

where $K = (2N - \nu\beta^2)/\nu$.

The second term in (26) or the expression (28) is a transient disturbance which decays to zero as $t \to +\infty$. If $\nu^2(\beta^2 + \alpha_n^2)^2 < 4N^2$, α_n corresponds to damped oscillatory transient disturbance, whereas if $\nu^2(\beta^2 + \alpha_n^2)^2 > 4N^2$, α_n corresponds to damped aperiodic motion. The first term in (26) represents a steady-state oscillation with the same frequency as that of the oscillation of the cylinder but with a phase difference ϕ given by

$$\phi = \arg [I_0(r\sqrt{L})/I_0(a\sqrt{L})]. \quad \ldots(29)$$

From (29), we find that, for any r ($< a$), ϕ is negative or positive according as ω^2 is greater than or less than N^2. If the frequency of oscillation of the cylinder is equal to the Brunt-Vaisälä frequency, then (26) and (28) become

$$w(r, t) = \sin(\omega t) \,.\, \frac{I_0(r\beta)}{I_0(a\beta)} + \frac{2\nu\omega}{a} \sum_{n=1}^{\infty} \frac{\exp(s_n t)\, J_0(r\alpha_n) \,.\, \alpha_n s_n^2}{(s_n^4 - \omega^4) \,.\, J_1(a\alpha_n)} \quad \ldots(30)$$

and

$$= -\frac{\nu}{a} \frac{\sqrt{R}\, J_0(r\sqrt{R}) \,.\, (t \exp(-\omega t))}{J_1(a\sqrt{R})} \quad \ldots(31)$$

where $R = (2\omega - \nu\beta^2)/\nu$.

The first term of (30) shows that, in the steady-state oscillation the flow is everywhere in phase with the oscillating boundary for all non-zero values of kinematic viscosity if $\omega = N$. Hence the special frequency $\omega = N$ may be called the natural frequency.

If the fluid is homogeneous, then $\beta = 0$, and (26) becomes

$$w(r,t) = \frac{1}{\lambda}(\sin \omega t\, [ber\, (r\sqrt{(\omega/\nu)})\, ber\, (a\sqrt{(\omega/\nu)})$$

$$+ bei\, (r\sqrt{(\omega/\nu)})\, bei\, (a\sqrt{(\omega/\nu)})]$$

$$+ \cos \omega t\, [bei\, (r\sqrt{(\omega/\nu)})\, ber\, (a\sqrt{(\omega/\nu)})$$

$$- ber\, (r\sqrt{(\omega/\nu)})\, bei\, (a\sqrt{(\omega/\nu)})])$$

$$+ \frac{2\nu\omega}{a} \sum_{n=1}^{\infty} \frac{\exp(-\nu\alpha_n^2 t)\, \alpha_n J_0(r\alpha_n)}{(\nu^2\alpha_n^4 + \omega^2)\, J_1(a\alpha_n)} \quad \ldots(32)$$

where $\lambda = ber^2\, (a\sqrt{(\omega/\nu)}) + bei^2\, (a\sqrt{(\omega/\nu)})$.

If the radius a of the cylinder is large, at all points near the boundary (where r/a is not small) (23) can be approximated to

$$w(r,t) \sim \frac{\omega}{2\pi i}(a/r)^{1/2} \int_{Br} \frac{\exp[st - (a-r)\,q]\, ds}{(s^2 + \omega^2)} \quad \ldots(33)$$

where

$$q = [(s^2 + \nu\beta^2 s + N^2)/\nu s]^{1/2}.$$

The steady-state solution in this case (as $t \to +\infty$) is

$$w(r,t) \sim (a/r)^{1/2} \exp(-\cos[\tfrac{1}{2}(\theta - \tfrac{1}{2}\pi)]\, A(a-r))$$

$$\times \sin(\omega t - \sin((1/2)(\theta - \tfrac{1}{2}\pi))\, A(a-r)) \quad \ldots(34)$$

where

$$A = (\beta^4 + [(\omega^2 - N^2)/\nu\omega]^2)^{1/4}$$

and

$$\tan\theta = -\frac{\omega\nu\beta^2}{\omega^2 - N^2}, \quad (0 < \theta \leqslant \pi).$$

The solution (26) obtained in the form of series converges more rapidly the greater the value of time. But it is not suitable for use with small values of time since the series is slowly convergent for such values. In this case, we replace the Bessel functions in (23) by their asymptotic values, e.g., $I_0(z) \sim \exp(z)/\sqrt{(2\pi z)}$.

Then

$$w(r,t) \sim \frac{\omega}{2\pi i}(a/r)^{1/2} \int_{Br} \frac{\exp[st - (a-r)\sqrt{(s+\nu\beta^2)}/\sqrt{\nu}]\, ds}{s^2}$$

(*equation continued on p.* 1114)

$$= \frac{\omega}{2} (a/r)^{1/2} \int_0^t [\exp(-(a-r)\beta) \operatorname{erfc}\{(a-r)/2\sqrt{(\nu\tau)} - \beta\sqrt{(\nu\tau)}\}$$

$$+ \exp((a-r)\beta) \operatorname{erfc}\{(a-r)/2\sqrt{(\nu\tau)} + \beta\sqrt{(\nu\tau)}\}] \, d\tau \qquad ...(35)$$

where r/a is not small.

The skin friction on the surface of the cylinder is given by the formula

$$\Lambda = \mu_0(z) [\partial w/\partial r]_{r=a} \qquad ...(36)$$

where w is given by (26). However, this is not suitable for small values of time. So, we shall obtain skin friction as follows (for small time). Differentiating (23) with respect to r and taking the limit as $r \to a$, we obtain the skin friction as the inversion integral

$$\Lambda = \frac{\mu_0(z)}{2\pi i} \int_{Br} \frac{\exp(st)\, qI_1(aq)\, ds}{(s^2+\omega^2)\, I_0(aq)}.$$

Using the asymptotic expansions for I_1 and I_0 for large s, we obtain the skin friction as

$$\Lambda \sim \frac{\omega\mu_0(z) \,.\, \sqrt{t}}{\sqrt{\pi}\,\sqrt{\nu}} \left(2 - \frac{\pi^{1/2}}{2}\left(\frac{\nu t}{a^2}\right)^{1/2} + \frac{(8\beta^2 a^2+1)}{12}\left(\frac{\nu t}{a^2}\right)\cdots\right). \qquad ...(37)$$

The above result clearly indicates that for very small values of $T = \nu t/a^2$ (i.e., when the diffusion length $\sqrt{(\nu t)}$ is much less than the radius of curvature a) the effects of surface curvature and perturbation in density are not significant. The effect of density perturbation appears from the third term onwards in the above expansion.

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