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ROTATIONS IN SPACE-TIME

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Corresponding to three timelike and three spacelike rotations in space-time, analogous to the Eulerian angles in the three dimensional space, the generators of the Lorentz group are obtained and thereby the changes in the components of a vector are calculated. Furthermore, the irreducible component of the representation space of rotation group is discussed.

1. INTRODUCTION

The orientation of a rigid body is specified by three independent parameters in 3-space. These parameters are the three successive angles of rotation about the axes and are termed as Eulerian angles. In extending the idea of rotation in four dimensional space-time of special relativity we need six parameters. Accordingly we transform an orthogonal tetrad into another in six steps i.e. through six successive rotations (not all real) and each rotation is about 2-flat instead of about an axis in 3-space (Synge 1972). We associate a set of six successive rotations to a Lorentz transformation (LT) because we know that any LT of the type $x'_i = A_{ij} x_j$ can be thought of as a succession of spacelike rotation

$$\left.\begin{aligned} x_1 &= x \cos\theta + y \sin\theta \\ y_1 &= -x \sin\theta + y\cos\theta,\ z_1 = z,\ t_1 = t \end{aligned}\right\} \quad ...(1.1)$$

through an angle θ in the 2-flat (x, y) and a timelike rotation

$$\left.\begin{aligned} x_1 &= x,\ y_1 = y,\ z_1 = z\cosh\phi + i\,t\sinh\phi \\ t_1 &= -i\,z\sinh\phi + t\cosh\phi,\ i^2 = -1 \end{aligned}\right\} \quad ...(1.2)$$

through a pseudo angle ϕ in the 2-flat (z, t). In this paper we shall consider three successive timelike rotations of the type (1.2), with parameters (ϕ_1, ϕ_2, ϕ_3), followed by three successive spacelike rotations of the type (1.1) with parameters $(\theta_1, \theta_2, \theta_3)$. These are finite rotations and association of a vector with the finite rotation does not work successfully However, the commutative property of infinitesimal transformations removes the objection to their representation by vectors. With this motivation we shall construct the generators (matrices of the infinitesimal transformation) of the Lorentz group by following Smirnov (1961). If a linear transformation is given by

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$$X' = G_p X \qquad \ldots(1.3)$$

where G_p is the matrix corresponding to the parameter p and X, X' are vectors, then an infinitesimal transformation of the group G are defined by

$$I_k = \left(\frac{\partial G_p}{\partial p_k}\right)_{p_j=0} \qquad \ldots(1.4)$$

The structure constants C^r_{pq} are then determined by

$$I_q I_p - I_p I_q = C^r_{pq}\, I_r. \qquad \ldots(1.5)$$

In section 2 are constructed the generators of the transformation. The structure constants and the changes in the components of a vector due to an infinitesimal transformation are calculated in section 3 and the representation of the rotation group has been discussed in section 4.

2. Spacelike and Timelike Rotations

Let us consider a transformation $(x, y, z, t) \to (x_6, y_6, z_6, t_6)$, which is carried out in six steps. The first three are the timelike rotations:

(a) $(x, y, z, t) \to (x_1, y_1, z_1, t_1)$ through pseudo angle ϕ_1 about 2-flat (y, z),

(b) $(x_1, y_1, z_1, t_1) \to (x_2, y_2, z_2, t_2)$ through pseudo angle ϕ_2 about 2-flat (z_1, x_1) and

(c) $(x_2, y_2, z_2, t_2) \to (x_3, y_3, z_3, t_3)$ through pseudo angle ϕ_3 about 2-flat (x_2, y_2).

One can easily write down the respective transformation matrices for (a), (b) and (c) with the help of (1.2). Then using eqn. (1.4), the generators of the three timelike rotations are given by

$$I_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$I_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}.$$

Next will be the three successive spacelike rotations as follows:

(d) $(x_3, y_3, z_3, t_3) \to (x_4, y_4, z_4, t_4)$ through an angle θ_1 about 2-flat (z_3, t_3),

(e) $(z_4, y_4, z_4, t_4) \to (x_5, y_5, z_5, t_5)$ through an angle θ_2 about 2-flat (x_4, t_4) and

(f) $(x_5, y_5, z_5, t_5) \to (x_6, y_6, z_6, t_6)$ through an angle θ_3 about 2-flat (y_5, t_5).

The corresponding infinitesimal matrices will be

$$I_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad I_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$I_6 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Structure Constants

An elementary calculation gives the following fifteen commutation relations:

$$\left.\begin{aligned} &I_1I_2 - I_2I_1 = I_4,\; I_1I_3 - I_3I_1 = -I_6,\; I_1I_4 - I_4I_1 = I_2 \\ &I_1I_5 - I_5I_1 = 0,\; I_1I_6 - I_6I_1 = -I_3,\; I_2I_3 - I_3I_2 = I_5 \\ &I_2I_4 - I_4I_2 = -I_1,\; I_2I_5 - I_5I_2 = I_3,\; I_2I_6 - I_6I_2 = 0 \\ &I_3I_4 - I_4I_3 = 0,\; I_3I_5 - I_5I_3 = -I_2,\; I_3I_6 - I_6I_3 = I_1 \\ &I_4I_5 - I_5I_4 = -I_6,\; I_4I_6 - I_6I_4 = I_5,\; I_5I_6 - I_6I_5 = -I_4. \end{aligned}\right\} \quad \ldots(3.1)$$

In view of (1.5) and (3.1), the structure constants can easily be determined and they assume the values 0, $\pm$ 1. Now expanding the right-hand side of eqn. (1.3) in powers of p_k and retaining only first order terms, the vector X changes by an amount

$$\delta X = (\phi_1 I_1 + \phi_2 I_2 + \phi_3 I_3 + \theta_1 I_4 + \theta_2 I_5 + \theta_3 I_6)\, X. \quad \ldots(3.2)$$

Therefore as a result of a rotation about the 2-flat (y, z) through the small angle ϕ_1, the changes in the components of X are

$$\delta X_1 = i\phi_1 X_4,\; \delta X_2 = 0,\; \delta X_3 = 0,\; \delta X_4 = -i\phi_1 X_1. \quad \ldots(3.3)$$

Similarly the result of successive rotations imply

$$\left.\begin{aligned} &\phi_2 : \delta X_1 = 0,\; \delta X_2 = i\phi_2 X_4,\; \delta X_3 = 0,\; \delta X_4 = -i\phi_2 X_2 \\ &\phi_3 : \delta X_1 = 0,\; \delta X_2 = 0,\; \delta X_3 = i\phi_3 X_4,\; \delta X_4 = -i\phi_3 X_3 \\ &\theta_1 : \delta X_1 = \theta_1 X_2,\; \delta X_2 = -\theta_1 X_1,\; \delta X_3 = 0,\; \delta X_4 = 0 \\ &\theta_2 : \delta X_1 = 0,\; \delta X_2 = \theta_2 X_3,\; \delta X_3 = -\theta_2 X_2,\; \delta X_4 = 0 \\ &\theta_3 : \delta X_1 = -\theta_3 X_3,\; \delta X_2 = 0,\; \delta X_3 = \theta_3 X_1,\; \delta X_4 = 0. \end{aligned}\right\} \quad \ldots(3.4)$$

One can see, from eqns. (3.3) and (3.4), that the first three rotations are imaginary as expected. Furthermore we note that the generators obey the following relations.

$$I_q^2 I_p^2 - I_p^2 I_q^2 = 0 \text{ and } I_p^3 = \pm I_p$$

where + sign is for timelike rotation and − sign for spacelike rotation.

4. Representation of the Rotation Group

We, now introduce the matrices

$$P_1 = I_3 - I_6 - i(I_2 + I_4),\ Q_1 = -I_3 - I_6 + i(I_2 - I_4)$$
$$P_2 = -I_3 - I_6 - i(I_2 - I_4),\ Q_2 = I_3 - I_6 + i(I_2 + I_4)$$
$$P_3 = (I_1 + iI_5)/2,\ Q_3 = (-I_1 + iI_5)/2.$$

It can easily be seen that these matrices satisfy the commutation relations

$$\left.\begin{aligned} &[P_1, P_2] = 8P_3,\ [Q_1, Q_2] = 8Q_3 \\ &[P_3, P_1] = P_1,\ [Q_3, Q_1] = Q_1 \\ &[P_3, P_2] = -P_2,\ [Q_3, Q_2] = -Q_2 \\ &[P_r, Q_r] = 0 \text{ for } r = 1, 2, 3 \end{aligned}\right\} \qquad \ldots(4.1)$$

where $[P, Q] = PQ - QP$.

We find that the relations (4.1) are similar to that of Smirnov's (1961) and hence his arguments can be applied here.

The operator P_3 has two distinct eigenvalues $\pm \frac{1}{2}$. Let the eigenvector of P_3 corresponding to the largest eigenvalue be v_j where $j = \frac{1}{2}$. By Smirnov's (1961) lemma

$$P_1 v_j = 0 \text{ and } P_2 v_j = v_{j-1}$$

be nonzero eigenvector of the operator P_3 corresponding to the eigenvalue $j - 1$. It can be proved, following Smirnov's (1961) that

$$P_1 v_k = A_k v_{k+1},\ k = j, j - 1$$

where A_k is an integer defined by the recursion formula

$$A_{k-1} = A_k + 8k,\ A_j = 0\ (k = j, j - 1, \ldots)$$

whose solution is

$$A_k = 4\,[j(j + 1) - k(k + 1)]$$
$$P_1 v_k = 4\,[j(j + 1) - k(k + 1]\ v_{k+1},\ (k = j, j - 1, \ldots).$$

Therefore, the nonzero vectors v_j and v_{-j} span a subspace, L_2, of V_4, the space of ordered quadruples. The operators P_1, P_2 and P_3 map the subspace L_2 into itself*.

*Vectors corresponding to the eigenvalue $+\frac{1}{2}$ of the operator P_3 are of the type $(x_1, x_2, -ix_2, ix_1)$ whereas corresponding to the eigenvalue $-\frac{1}{2}$ are of the type $(x_1, x_2, ix_2, -ix_1)$.

This subspace L_2, of the representation space of rotations, is an irreducible component of dimension two.

The vectors v_j and v_{-j} can be multiplied by arbitrary nonzero constants. The constants can be chosen in such a way that we have

$$\left.\begin{aligned} P_1 v_k &= \sqrt{j(j+1) - k(k+1)}\, v_{k+1} \\ P_2 v_k &= \sqrt{j(j+1) - k(k-1)}\, v_{k-1}, \ (k = j, -j) \\ P_3 v_k &= k v_k \end{aligned}\right\} \quad \ldots(4.2)$$

where $v_{j+1} = 0$ and $v_{-j-1} = 0$.

If the vector v belongs to L_2, $Q_r v$ $(r = 1, 2, 3)$ also belongs to L_2. In fact, we have

$$P_3(Q_r v) = Q_r P_3 v = Q_r j v = j Q_r v$$

so that $Q_r v$ is an eigenvector of P_3 corresponding to the eigenvalue j. Replacing the operators P_r by the operators Q_r, we can construct vectors in L_2, $v_{jk'}$, $(k' = \frac{1}{2}, -\frac{1}{2})$ which are transformed by the similar formulae obtained from (4.2) by replacing P_r by Q_r.

Thus we obtain four vectors $v_{kk'}$ which satisfy the relations

$$\begin{aligned} P_1 v_{kk'} &= \sqrt{j(j+1) - k(k+1)}\ v_{k+1,\,k'} \\ P_2 v_{kk'} &= \sqrt{j(j+1) - k(k-1)}\ v_{k-1,\,k'} \\ P_3 v_{kk'} &= k v_{kk'} \\ Q_1 v_{kk'} &= \sqrt{j(j+1) - k(k+1)}\ v_{k,\,k'+1} \\ Q_2 v_{kk'} &= \sqrt{j(j+1) - k(k-1)}\ v_{k,\,k'-1} \\ Q_3 v_{kk'} &= k' v_{kk'}. \end{aligned}$$

These relations define the operators P_r and Q_r in a direct product space of dimension four.

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