Indian J. pure appl. Math., **10**(9) : 1121-1126, September 1979

ON THE KONHAUSER POLYNOMIALS $Y_n^\alpha(x;k)$

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(*Received* 19 *September* 1978; *after revision* 30 *January* 1979)

The Rodrigues' formula for the Konhauser polynomials $Y_n^\alpha(x;k)$ has been used to obtain an operational formula and a generating relation for these polynomials. Several particular cases of interest have been noticed.

1. INTRODUCTION

Recently, Konhauser (1967) discussed two polynomials $\{Z_n^\alpha(x;k)\}$ and $\{Y_n^\alpha(x;k)\}$, biorthogonal with respect to the weight function $x^\alpha \exp(-x)$ over the interval $(0,\infty)$, $\alpha > -1$ and k being a positive integer. These polynomials satisfy the following condition:

$$\int_0^\infty x^\alpha \exp(-x)\, Y_i^\alpha(x;k)\, Z_j^\alpha(x;k)\, dx \begin{cases} = 0, & i \neq j, \\ \neq 0, & i = j; \end{cases}$$

$$i, j = 0, 1, 2, \ldots .$$

For $k = 1$, the polynomials $Z_n^\alpha(x;k)$ as well as $Y_n^\alpha(x;k)$ reduce to the Laguerre polynomials $L_n^\alpha(x)$.

In the same paper Konhauser obtained mixed recurrence relations and differential equations for this set of polynomials. The study of these polynomials was further continued by Prabhakar (1970, 1971), Srivastava (1973) and Srivastava and Singh (1978). Prabhakar (1971) obtained, among other things, the following Rodrigues' formula for $Y_n^\alpha(x;k)$:

$$Y_n^\alpha(x;k) = \frac{e^x x^{k-\alpha-1}}{n!}\, [D_s^n\, \{s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})\}]_{s=x^k}\,. \qquad \ldots(1.1)$$

Indeed, the relation (1.1) follows at once from the following relation due to Srivastava and Lavoie [1975, p. 315, eqn. (83)]

$$Y_n^\alpha(x;k) = k^{-n} G_n^{(\alpha+1)}(x, 1, 1, k), \quad \alpha > -1, k = 1, 2, 3, \ldots \qquad \ldots(1.2)$$

and the Rodrigues' representation of $G_n^{(\alpha)}(x, r, p, k)$, viz (see Srivastava and Singhal [1971, p. 75, eqn. (1.3)]):

$$G_n^{(\alpha)}(x, r, p, k) = \frac{1}{n!} x^{-\alpha-kn} \exp(px^r)\, \theta_x^n \{x^\alpha \exp(-px^r)\} \qquad \ldots(1.3)$$

where $\theta_x \equiv x^{k+1}D_x$ and $\{G_n^{(\alpha)}(x, r, p, k) \mid n = 0, 1, 2, \ldots\}$

is the class of polynomials introduced earlier by Srivastava and Singhal (1971).

The aim of the present investigation is to establish an operational formula and a generating relation for $Y_n^\alpha(s^{1/k}; k)$ by using the formula (1.1).

2. An Operational Formula

The following well-known properties of the operator $\delta_x \equiv xD_x$ will be used in our analysis.

$$x^n D_x^n = \delta_x(\delta_x - 1) \ldots (\delta_x - n + 1) \qquad \ldots(2.1)$$

$$f(\delta_x)\{\exp(g(x))\, h(x)\} = \exp(g(x))\, f(\delta_x + xg'(x))\{h(x)\} \qquad \ldots(2.2)$$

$$f(\delta_x)\{x^\alpha h(x)\} = x^\alpha f(\delta_x + \alpha)\, h(x) \qquad \ldots(2.3)$$

and

$$x^{n\alpha} f(\delta_x)\, f(\delta_x + \alpha) \ldots f(\delta_x + (n-1)\alpha) = [x^\alpha f(\delta_x)]^n \qquad \ldots(2.4)$$

for every integer $n \geqq 1$.

From (1.1), we have

$$Y_n^\alpha(s^{1/k}; k) = \frac{\exp(s^{1/k})\, s^{1-(\alpha+1)/k}}{n!} D_s^n [s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})], \qquad \ldots(2.5)$$

where $D_s \equiv d/ds$.

Consider

$$\exp(s^{1/k})\, s^{1-(1+\alpha)/k} D_s^n [s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})\, F]$$

$$= \exp(s^{1/k})\, s^{1-n-(\alpha+1)/k} \delta_s(\delta_s - 1) \ldots (\delta_s - n + 1)$$

$$\times [s^{n-1+(\alpha+1)/k} \exp(-s^{1/k})\, F]$$

$$= \exp(s^{1/k})\, s^{1-c-(\alpha+1)/k}(\delta_s - c + 1)(\delta_s - c + 2) \ldots (\delta_s - c + n)$$

$$\times [s^{c-1+(\alpha+1)/k} \exp(-s^{1/k})\, F].$$

Now, the right side of the above expression, in view of relation (2.2), becomes

$$\exp((1+\beta)s^{1/k})\, s^{1-c-(\alpha+1)/k}\left(\delta_s - c + \frac{1}{k}s\beta s^{(1-k)/k} + 1\right)$$
$$\times\left(\delta_s - c + \frac{1}{k}s\beta s^{(1-k)/k} + 2\right)\ldots\left(\delta_s - c + \frac{1}{k}s\beta s^{(1-k)/k} + n\right)$$
$$\times [s^{c-1+(\alpha+1)/k}(\exp(-\beta-1)s^{1/k})F].$$

We have, therefore, proved that

$$D_s^n[s^{n-1+(\alpha+1)/k}\exp(-s^{1/k})F]$$
$$= \exp(\beta s^{1/k})\, s^{-c}\prod_{j=1}^{n}\left(\delta_s - c + \frac{s\beta s^{(1-k)/k}}{k} + j\right)$$
$$\times [s^{c-1+(\alpha+1)/k}(\exp(-\beta-1)s^{1/k})F]. \qquad \ldots(2.6)$$

On the other hand, it is not difficult to establish the following relation by making use of Leibnitz rule of differentiation,

$$D_s^n[s^{n-1+(\alpha+1)/k}\exp(-s^{1/k})F]$$
$$= \exp(-s^{1/k})\, s^{-1+(\alpha+1)/k}\sum_{p=0}^{n}\frac{n!\, s^p}{p!}Y_{n-p}^{\alpha+kp}(s^{1/k}; k)\, D_s^p F. \qquad \ldots(2.7)$$

Therefore, in view of (2.6) and (2.7), we are led to the operational formula

$$\prod_{j=1}^{n}\left(\delta_s - c + \frac{1}{k}s\beta s^{(1-k)/k} + j\right)[s^{c-1+(\alpha+1)/k}(\exp(-\beta-1)s^{1/k})F]$$
$$= n!\exp\{(-\beta-1)s^{1/k}\}\, s^{c-1+(\alpha+1)/k}$$
$$\times\sum_{p=0}^{n}\frac{s^p}{p!}Y_{n-p}^{\alpha+kp}(s^{1/k}; k)\, D_s^p F \qquad \ldots(2.8)$$

where $\quad \delta_s \equiv sD_s.$

3. Particular Cases

(i) First we note in particular that, for $k = 1$, (2.6) becomes

$$D_s^n[s^{\alpha+n}e^{-s}F] = e^{\beta s}s^{-c}\prod_{j=1}^{n}(\delta_s - c + \beta s + j)[s^{\alpha+c}(\exp(-\beta-1)s)F] \qquad \ldots(3.1)$$

a result due to Thakare and Karande (1973).

(ii) If we choose $k = 1$, $\beta = -1$ and $c = -\alpha$, then the formula (2.6) would correspond to Carlitz's result (1960)

$$D_s^n\,[s^{\alpha+n}e^{-s}F] = s^{\alpha}e^{-\alpha}\prod_{j=1}^{n}(\delta_s + \alpha - s + j)\,F. \qquad \text{...(3.2)}$$

(iii) The result due to Das (1967) can be obtained by taking $k = 1$ and $\beta = 0$ in (2.6).

(iv) Taking $k = 1$ in (2.7) we obtain

$$D_s^n\,[s^{\alpha+n}e^{-s}F] = e^{-s}s^{\alpha}n!\sum_{p=0}^{n}\frac{s^p}{p!}L_{n-p}^{\alpha+p}(s)\,D_s^p\,F \qquad \text{...(3.3)}$$

a result due to Carlitz (1960) for generalized Laguerre polynomials.

(v) The substitution $k = 1$ reduces (2.8) to the following result of Thakare and Karande (1973),

$$\prod_{i=1}^{n}(\delta_s - c + \beta s + j)\,[s^{\alpha+c}(\exp(-\beta-1)\,s)\,F]$$

$$= n!\,s^{\alpha+c}\exp\{(-\beta-1)\,s\}\sum_{p=0}^{n}\frac{s^p}{p!}L_{n-p}^{\alpha+p}(s)\,D_s^p\,F. \qquad \text{...(3.4)}$$

(vi) If in (2.8) we let $k = 1$, $\beta = c = 0$, we fall back to a known result due to Al-Salam (1964).

(vii) The operational formula (2.8) with $F = 1$ yields

$$\prod_{j=1}^{n}\left(\delta_s - c + \frac{1}{k}\,s\beta s^{(1-k)/k} + j\right)[s^{c-1+(\alpha+1)/k}\exp(-\beta-1)\,s^{1/k}]$$

$$= n!\exp[(-\beta-1)\,s^{1/k}]\,s^{c-1+(\alpha+1)/k}\,Y_n^{\alpha}(s^{1/k};k) \qquad \text{...(3.5)}$$

which does not seem to have appeared earlier.

4. Generating Relation

From formula (2.5) we note that

$$Y_n^{\alpha-kn}(s^{1/k};k) = \frac{1}{n!}\exp(s^{1/k})\,s^{n+1-(\alpha+1)/k}\,D_s^n\,[s^{(\alpha-k+1)/k}\exp(-s^{1/k})].$$

$$\text{...(4.1)}$$

Therefore

$$\sum_{n=0}^{\infty} s^{-n} Y_n^{\alpha-kn}(s^{1/k}; k)\, t^n$$

$$= \exp(s^{1/k})\, s^{-(\alpha-k+1)/k} e^{tD_s} [s^{(\alpha-k+1)/k} \exp(-s^{1/k})]$$

$$= \left(1 + \frac{t}{s}\right)^{(\alpha-k+1)/k} \exp\left[s^{1/k}\left\{1 - \left(1 + \frac{t}{s}\right)^{1/k}\right\}\right].$$

Thus we derive

$$\sum_{n=0}^{\infty} Y_n^{\alpha-kn}(s^{1/k}; k)\, t^n = (1 + t)^{(\alpha-k+1)/k} \exp[s^{1/k}\{1 - (1 + t)^{1/k}\}]. \quad \ldots(4.2)$$

In view of the relationship (1.2), the generating function (4.2) follows also as an obvious special case of a known generating function due to Srivastava and Singhal [(1971), p. 79, eqn. (3.4)] on setting $p = r = 1$ and $x = s^{1/k}$, and on replacing α by $\alpha + 1$ and t by kt. {In fact, the generating function (4.2) with $s = x^k$ is a well-known result.}

Alternatively, (4.2) can be established by using (4.1) and Taylor's formula in the form:

$$F[x(1 + t)] = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} D_x^n \{F(x)\}. \quad \ldots(4.3)$$

Acknowledgement

The authors are very thankful to the referee who drew their attention to a number of mistakes in the first version of this paper.

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