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TIME-DEPENDENT THERMAL SOLUTION FOR A SLAB

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Solution of a two-dimensional heat conduction problem is discussed keeping one face of the slab at constant temperature and exposing the other face to a variable temperature distribution or perscribed heat input which are functions of distance and time. This model assimilates the conditions encountered by missiles and other space vehicles.

1. INTRODUCTION

An interesting solution to heat conduction equation is obtained for an infinite slab when one face of the slab is kept at constant temperature and the other is exposed to a variable temperature distribution or prescribed heat input. This model was generated by Martin and Payton (1964) and Martin (1966) to approximate the temperature distribution and the elastic field in bonding materials used in missiles and space vehicles. They have solved this problem by employing the theory of complex variables and Fourier transforms when the applied surface temperature is a Heaviside unit step function. Here, the same model is taken to investigate the temperature distribution in the slab using a different method when the applied surface temperature or the prescribed heat input is any continuous or piecewise continuous function of the distance and time.

The method adopted by Lekhnitskii (1963) and Lur'e (1964) is used to solve the problem when the applied surface temperature or the prescribed heat input is a continuous or piecewise continuous function of the distance or time or both. The method makes use of the expansion of a class of functions $f(q)$ of the complex variable q which can be written in rational fractions (Mittag-Leffler 1880). The function $f(q)$ is such that its only singularities in the finite part of the plane are simple poles $a_1, a_2, a_3, \ldots$, where $|a_1| \leqslant |a_2| \leqslant |a_3| \ldots$. If $b_1, b_2, \ldots$ are the residues at these poles and let it be possible to choose a sequence of circles C_m (the radius of C_m being R_m) with centre at O, not passing through any poles, such that $|f(q)|$ is bounded on C_m, and $R_m \to \infty$ as $m \to \infty$, then the function $f(q)$ has the following expansion in rational fractions :

$$f(q) = f(0) + \sum_{n=1}^{\infty} \left[b_n \left(\frac{1}{q - a_n} \right) + \frac{b_n}{a_n} \right] \qquad \ldots(1.1)$$

where the summation extends over all the poles of $f(q)$. The temperature distribution is finally obtained, by using the result.

$$\frac{1}{q-a} f(t) = e^{at} \int_b^t f(\zeta)\, e^{-a\zeta} d\zeta, \quad q \equiv \frac{\partial}{\partial t} \qquad ...(1.2)$$

where the constant b is chosen to satisfy the initial conditions and t stands for time.

2. Statement of the Problem and Symbolic Solution

Let the infinite slab be of constant finite thickness h. The origin of the rectangular Cartesian co-ordinate system is taken on the lower face which is taken as the xz-plane. The temperature T in the slab, in this case, is assumed to satisfy the linear heat conduction equation :

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\rho c}{k} \frac{\partial T}{\partial t} \qquad ...(2.1)$$

where ρ, c, k are respectively, the density, specific heat and the thermal conductivity of the solid. The slab is assumed to be initially at zero temperature. The boundary conditions in this case are assumed as

$$\left.\begin{aligned} &T(x, 0, t) = 0, \quad \text{for all } x \text{ and } t \\ &T(x, h, t) = G(x, t), \end{aligned}\right] \qquad ...(2.2)$$

where $G(x, t)$ is a continuous or piecewise function of x and t.

Defining dimensionless quantities

$$x' = x/h,\ y' = y/h,\ t' = kt/\rho c h^2 \qquad ...(2.3)$$

the heat eqn. (2.1) and the thermal boundary conditions take the form :

$$\left.\begin{aligned} &\frac{\partial^2 T}{\partial x'^2} + \frac{\partial^2 T}{\partial y'^2} = \frac{\partial T}{\partial t'} \\ &T(x', 0, t') = 0,\ T(x', 1, t') = G'(x', t'). \end{aligned}\right\} \qquad ...(2.4)$$

The primes are omitted in what follows.

Writing $p \equiv \frac{\partial}{\partial x}$, $q \equiv \frac{\partial}{\partial t}$ the solution of the heat equation in the operator form is obtained in the following form (Wadhawan 1972, 1973) :

$$T(x, y, t) = \frac{\sin y \sqrt{p^2 - q}}{\sin \sqrt{p^2 - q}}\, G(x, t). \qquad ...(2.5)$$

3. Particular Cases

Let the surface temperature $G(x, t)$ on the face $y = 1$ of the slab be prescribed as

$$G(x, t) = T_0 e^{-b|x|} \phi(t), \quad -\infty \leqslant x \leqslant \infty \qquad \text{...(3.1)}$$

where $\phi(t)$ is continuous or piecewise continuous function of time and b is any constant such that $\operatorname{Re} b > 0$.

Writing $F(p, q) = \dfrac{\sin y\sqrt{p^2 - q}}{\sin \sqrt{p^2 - q}}$ and using (2.5) and (3.1), we obtain

$$T(x, y, t) = T_0 e^{-b|x|} F(-b, q)\, \phi(t), \quad x > 0$$

$$T(x, y, t) = T_0 e^{-b|x|} F(b, q)\, \phi(t), \quad x < 0$$

and since $F(p, q)$ is an even function in p, we have

$$T(x, y, t) = T_0 e^{-b|x|} F(b, q)\, \phi(t), \quad -\infty \leqslant x \leqslant \infty. \qquad \text{...(3.2)}$$

The function $F(b, q)$ satisfies the conditions of Mittag-Leffler Theorem and $F(b, q)$ has simple poles at

$$q = b^2 - n^2\pi^2, \quad n = 1, 2, 3, \dots . \qquad \text{...(3.3)}$$

Evaluating the residues at the simple poles and making use of (1.1) and (1.2), the temperature distribution is obtained as :

$$T(x, y, t) = T_0 e^{-b|x|} \left[\frac{\sin yb}{\sin b} \phi(t) + \sum_{n=1}^{\infty} 2n\pi(-1)^{n+1} \sin n\pi y \right.$$
$$\left. \times \left\{ e^{(b^2 - n^2\pi^2)t} \int_0^t e^{-(b^2 - n^2\pi^2)\tau} \phi(\tau)\, d\tau + \frac{\phi(t)}{b^2 - n^2\pi^2} \right\} \right]. \qquad \text{...(3.4)}$$

Example 1 — The function $\phi(t)$ can be prescribed and the integral in (3.4) can be evaluated. As an example consider the case of sudden heating. We, thus, prescribe $\phi(t) = H(t)$, where $H(t)$ is the Heaviside unit step function defined as

$$\left.\begin{aligned} H(t) &= 0, \; t < 0 \\ &= 1, \; t > 0. \end{aligned}\right\} \qquad \text{...(3.5)}$$

The expression for temperature distribution takes the form

$$T(x, y, t) = T_0 e^{-b|x|} \left[\frac{\sin yb}{\sin b} H(t) \right.$$
$$\left. + \sum_{n=1}^{\infty} \left\{ (-1)^{n+1} \frac{2n\pi \sin n\pi y}{b^2 - n^2\pi^2} (e^{b^2 - n^2\pi^2} - 1) + \frac{H(t)}{b^2 - n^2\pi^2} \right\} \right]. \qquad \text{...(3.6)}$$

Example 2 — Take the case when the surface temperature $G(x, t)$ is prescribed as

$$G(x, t) = T_0 t e^{-ct} e^{-b|x|}. \qquad ...(3.7)$$

In this case $\phi(t) = te^{-ct}$, where c is a constant such that Re $c > 0$.

The temperature distribution $T(x, y, t)$ is obtained as

$$T(x, y, t) = T_0 \Bigg[t e^{-b|x|} e^{-ct} \frac{\sin yb}{\sin b} + e^{-b|x|} \sum_{n=1}^{\infty} (-1)^{n+1} 2n\pi y \sin n\pi y$$
$$\times \left\{ \frac{1}{a^2} e^{(b^2-n^2\pi^2)t}(1 - (1 + at) e^{-at}) + \frac{e^{-b|x|}}{b^2 - n^2\pi^2} te^{-ct} \right\} \Bigg] \qquad ...(3.8)$$

where $a = b^2 + c - n^2\pi^2$.

It may be remarked that the initial condition $T(x, y, 0) = 0$ is satisfied.

4. Prescribed Heat Input

Let us now study the case when the slab is subjected to prescribed heat input at one face and the other face is kept at zero temperature.

The thermal boundary conditions in this case are :

$$\left. \begin{aligned} &T(x, 0, t) = 0, \quad \text{for all } x \text{ and } t \\ &k \frac{\partial T}{\partial y} = \bar{G}(x, t), \quad \text{at} \quad y = h \end{aligned} \right\} \qquad ...(4.1)$$

where k is the thermal conductivity of the slab and $\bar{G}(x, t)$ is a continuous or piecewise continuous function of x and t. As before, initial condition is prescribed as :

$$T(x, y, 0) = 0, \ -\infty \leqslant x \leqslant \infty, \quad 0 \leqslant y \leqslant 1. \qquad ...(4.2)$$

The solution of eqn. (2.4), on using the boundary conditions (4.1), can be written as

$$T(x, y, t) = \frac{\sin y \sqrt{p^2 - q}}{k \sqrt{p^2 - q} \cos \sqrt{p^2 - q}} \bar{G}(x, t). \qquad ...(4.3)$$

The heat input function $\bar{G}(x, t)$ can be prescribed as follows :

$$\bar{G}(x, t) = \bar{T}_0 e^{-b|x|} \psi(t) \qquad ...(4.4)$$

where $\bar{T}_0$ and b are constants and $\psi(t)$ is any function of t.

Writing $T = \bar{F}(p, q)\,\bar{G}(x, t)$, and using (4.3) and (4.4) we obtain

$$T(x, y, t) = \bar{T}_0 e^{-b|x|} \bar{F}(-b, q)\,\psi(t), \quad \text{for} \quad x > 0$$

and

$$T(x, y, t) = \bar{T}_0 e^{-b|x|} \bar{F}(b, q)\,\psi(t), \quad \text{for} \quad x < 0,$$

and since $F(p, q)$ is an even function in p, we have

$$T(x, y, t) = \bar{T}_0 e^{-b|x|} \bar{F}(b, q)\,\psi(t), \; -\infty \leqslant x \leqslant \infty.$$

The function $F(b, q)$ satisfies the conditions of Mittag-Leffler Theorem. The simple poles of $F(b, q)$ are at

$$q = b^2 - (2m+1)^2 (\pi^2/4) \qquad \ldots(4.5)$$

where $m = 0, 1, 2, 3, \ldots$.

Evaluating the residues at these poles, and applying (1.1) and (1.2), the temperature distribution is obtained :

$$T(x, y, t) = \frac{\bar{T}_0}{k} e^{-b|x|} \Bigg[\frac{\sin yb}{b \cos b} \psi(t) + 2 \sum_{m=0}^{\infty} (-1)^m \sin (2m+1)(\pi/2)\, y \Bigg\{ e^{(b^2-(2m+1)^2(\pi^2/4))} \times \int_0^t e^{-(b^2-(2m+1)^2(\pi^2/4))\tau} \psi(\tau)\, d\tau + \frac{\psi(t)}{(b^2-(2m+1)^2(\pi^2/4)} \Bigg\} \Bigg]. \qquad \ldots(4.6)$$

As an example, consider the case when $\psi(t)$ is a unit step function defined in (3.5).

The temperature distribution in this case is given by the expression :

$$T(x, y, t) = \frac{\bar{T}_0}{k} e^{-b|x|} \Bigg[\frac{\sin yb}{b \cos b} H(t) + 2 \sum_{m=1}^{\infty} \Bigg\{ (-1)^m \frac{\sin (2m+1)(\pi y/2)}{b^2-(2m+1)^2(\pi^2/4)} \times \left(e^{(b^2-(2m+1)^2(\pi^2/4))} - 1 \right) + \frac{H(t)}{b^2-(2m+1)^2(\pi^2/4)} \Bigg\} \Bigg].$$

The initial condition is obviously satisfied.

The method adopted in this paper proves to be useful for solution of a number of thermal problems. It can also be used for other type of boundary conditions e.g. radiation conditions at the boundary.

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