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A CLASS OF STATIONARY BRANS-DICKE VACUUM FIELDS

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Static axially symmetric Brans-Dicke (B-D) Maxwell fields, when both electric and magnetic potentials are simultaneously present, is considered. It is shown that, when a change in the sign of the gravitational constant k occurs, the above field equations for the case $\omega = -3/2$, go over to stationary B-D vacuum fields for $\omega = 0$.

INTRODUCTION

Perjes (1968) has constructed the stationary Einstein vacuum solutions from the static axially symmetric Einstein-Maxwell fields when both electric and magnetic potentials are simultaneously present. We, in this paper, have extended the above result to generate the stationary Brans-Dicke vaccum solutions for the case when the coupling constant $\omega = 0$, from the static axially symmetric Brans-Dicke Maxwell solutions for $\omega = -3/2$. In doing so, we have used the results of our previous paper (Rao and Tiwari 1978), wherein we have generated static axially symmetric B-D Maxwell fields, when both electric and magnetic potentials are simultaneously present, from the corresponding B-D vacuum solutions.

The physical importance of these solutions can be realized from the very fact that they belong to the well-known vacuum Kerr family of solutions of the B-D theory. The B-D coupling constant being equal to zero, however, is a very serious limilation on these solutions which deserves further investigation.

FIELD EQUATIONS AND SOLUTIONS

The source free B-D Maxwell field equations are

$$R_{ij} = -\frac{k}{\phi} T_{ij} - \frac{\omega}{\phi^2} \phi_{,i}\phi_{,j} - \frac{1}{\phi} \phi_{;ij} \qquad \ldots(1)$$

$$\phi_{;s}^{;s} = 0 \qquad \ldots(2)$$

$$T^i_j = -\frac{1}{4\pi} (F^{is} F_{js} - \tfrac{1}{4}\delta^i_j F^{sp} F_{sp}) \qquad \ldots(3)$$

$$F^{ij}_{;j} = 0 \qquad \ldots(4)$$

$$F_{ij} = A_{i,j} - A_{j,i} \qquad \ldots(5)$$

where k and A_i are the gravitational constant and the four potential respectively. Here and in what follows a subscript comma or a semicolon denote partial differentiation or covariant differentiation respectively.

The field eqns. (1) – (5) for the static axially symmetric Weyl's metric

$$ds^2 = e^{2\beta}dt^2 - e^{-2\beta}\left[e^{2\alpha}(d\rho^2 + dz^2) + \rho^2 d\Phi^2\right] \qquad ...(6)$$

where α and β are functions of ρ and z, are given as

$$\alpha_{,11} + \alpha_{,22} - \beta_{,11} - \beta_{,22} + 2\beta_{,1}^2 - \frac{\alpha_{,1}}{\rho} - \frac{\beta_{,1}}{\rho}$$
$$= \frac{k}{8\pi\phi}\left[-\frac{e^{2\beta}}{\rho^2}(\xi_{,1}^2 - \xi_{,2}^2) + e^{-2\beta}(\eta_{,1}^2 - \eta_{,2}^2)\right]$$
$$- \omega\frac{\phi_{,1}^2}{\phi^2} - \frac{1}{\phi}\left[\phi_{,11} - \phi_{,1}(\alpha_{,1} - \beta_{,1}) + \phi_{,2}(\alpha_{,2} - \beta_{,2})\right] \qquad ...(7)$$

$$\alpha_{,11} + \alpha_{,22} - \beta_{,11} - \beta_{,22} + 2\beta_{,2}^2 + \frac{\alpha_{,1}}{\rho} - \frac{\beta_{,1}}{\rho}$$
$$= \frac{k}{8\pi\phi}\left[\frac{e^{2\beta}}{\rho^2}(\xi_{,1}^2 - \xi_{,2}^2) - e^{-2\beta}(\eta_{,1}^2 - \eta_{,2}^2)\right]$$
$$- \omega\frac{\phi_{,2}^2}{\phi^2} - \frac{1}{\phi}\left[\phi_{,22} + \phi_{,1}(\alpha_{,1} - \beta_{,1}) - \phi_{,2}(\alpha_{,2} - \beta_{,2})\right] \qquad ...(8)$$

$$\beta_{,11} + \beta_{,22} + \frac{\beta_{,1}}{\rho} = \frac{k}{8\pi\phi}\left[\frac{e^{2\beta}}{\rho^2}(\xi_{,1}^2 + \xi_{,2}^2) + e^{-2\beta}(\eta_{,1}^2 + \eta_{,2}^2)\right]$$
$$+ \frac{\phi_{,1}}{\phi}\left(\frac{1}{\rho} - \beta_{,1}\right) - \frac{\phi_{,2}}{\phi}\beta_{,2} \qquad ...(9)$$

$$\beta_{,11} + \beta_{,22} + \frac{\beta_{,1}}{\rho^2} = \frac{k}{8\pi\phi}\left[\frac{e^{2\beta}}{\rho^2}(\xi_{,1}^2 + \xi_{,2}^2) + e^{-2\beta}(\eta_{,1}^2 + \eta_{,2}^2)\right]$$
$$- \frac{1}{\phi}(\phi_{,1}\beta_{,1} + \phi_{,2}\beta_{,2}) \qquad ...(10)$$

$$2\beta_{,1}\beta_{,2} - \frac{\alpha_{,2}}{\rho} = \frac{k}{4\pi\phi}\left[\frac{e^{2\beta}}{\rho^2}\xi_{,1}\xi_{,2} + e^{-2\beta}\eta_{,1}\eta_{,2}\right] - \frac{\omega}{\phi^2}\phi_{,1}\phi_{,2}$$
$$- \frac{1}{\phi}\left[\phi_{,12} - \phi_{,1}(\alpha_{,2} - \beta_{,2}) - \phi_{,2}(\alpha_{,1} - \beta_{,1})\right] \qquad ...(11)$$

$$\xi_{,1}\eta_{,1} - \xi_{,2}\eta_{,2} = 0 \qquad ...(12)$$

$$\xi_{,11} + \xi_{,22} - \frac{\xi_{,1}}{\rho} = -2\beta_{,1}\xi_{,1} - 2\beta_{,2}\xi_{,2} \qquad ...(13)$$

$$\eta_{,11} + \eta_{,22} + \frac{\eta_{,1}}{\rho} = 2\beta_{,1}\eta_{,1} + 2\beta_{,2}\eta_{,2} \qquad ...(14)$$

$$\phi_{,11} + \phi_{,22} + \frac{\phi_{,1}}{\rho} = 0. \qquad ...(15)$$

Equations (9) and (10) together give $\phi_{,1} = 0$, which when substituted in eqn. (15) gives

$$\phi_{,2} = \text{constant} = m \text{ (say)} \qquad ...(16)$$

i.e.,
$$\phi = mz + n \qquad ...(17)$$

where n is an integration constant.

Substituting (Tauber 1957)

$$\xi_{,1}\frac{e^{2\beta}}{\rho} = \psi_{,2}, \quad \xi_{,2}\frac{e^{2\beta}}{\rho} = -\psi_{,1} \qquad ...(18)$$

in (13), we get

$$\psi_{,11} + \psi_{,22} + \frac{\psi_{,1}}{\rho} = 2\beta_{,1}\psi_{1} + 2\beta_{,2}\psi_{,2}. \qquad ...(19)$$

We now assume η to be dependent on ψ, viz.,

$$\eta = \eta(\psi). \qquad ...(20)$$

Using (20) in (14) and (19), we readily obtain

$$\eta = a\psi + b \qquad ...(21)$$

where a and b are arbitrary constants of integration.

Relations (17), (18) and (21) when used in eqns. (7) – (14) and (19), give

$$\alpha_{,1}\left(\frac{1}{\rho^2} + \frac{\phi_{,2}^2}{\phi^2}\right) = \frac{1}{\rho}\left(\beta_{,1}^2 - \beta_{,2}^2\right) + 2\beta_{,1}\beta_{,2}\frac{\phi_{,2}}{\phi} - \frac{\omega}{2}\frac{\phi_{,2}^2}{\rho\phi^2} - \frac{\phi_{,2}}{\phi}\left(\frac{\beta_{,2}}{\rho} - \beta_{,1}\frac{\phi_{,2}}{\phi}\right) + \frac{k}{8\pi\phi}\frac{e^{2\beta}}{\rho^3}(1+a^2)\left(\xi_{,1}^2 - \xi_{,2}^2\right) + \frac{k}{4\pi}\frac{\phi_{,2}}{\phi^2}\frac{e^{2\beta}}{\rho^2}(1+a^2)\,\xi_{,1}\xi_{,2} \qquad ...(22)$$

$$\alpha_{,2}\left(\frac{1}{\rho^2} + \frac{\phi_{,2}^2}{\phi^2}\right) = -\frac{\phi_{,2}}{\phi}\left(\beta_{,1}^2 - \beta_{,2}^2\right) + \frac{2\beta_{,1}\beta_{,2}}{\rho} + \frac{\omega}{2}\frac{\phi_{,2}^3}{\phi^3} + \frac{\phi_{2}}{\phi}\left(\frac{\beta_{,2}\phi_{,2}}{\phi} + \frac{\beta_{,1}}{\rho}\right) + \frac{k}{4\pi\phi}\frac{e^{2\beta}}{\rho^3}(1+a^2)\,\xi_{,1}\xi_{,2} - \frac{k}{8\pi}\frac{\phi_{,2}}{\phi^2}\frac{e^{2\beta}}{\rho^2}(1+a^2)\left(\xi_{,1}^2 - \xi_{,2}^2\right) \qquad ...(23)$$

$$\beta_{,11} + \beta_{,22} + \frac{\beta_{,1}}{\rho} = \frac{k}{8\pi\phi}\frac{e^{2\beta}}{\rho^2}(1+a^2)\left(\xi_{,1}^2 + \xi_{,2}^2\right) - \frac{\beta_{,2}\phi_{,2}}{\phi} \qquad ...(24)$$

$$\xi_{,11} + \xi_{,22} - \frac{\xi_{,1}}{\rho} = -2\beta_{,1}\xi_{,1} - 2\beta_{,2}\xi_{,2}. \quad \ldots(25)$$

Equations (22) – (25), using (18), can further be expressed in the following equivalent form:

$$\alpha_{,1}\left(\frac{1}{\rho^2} + \frac{\phi_{,2}^2}{\phi^2}\right) = \frac{1}{\rho}(\beta_{,1}^2 - \beta_{,2}^2) + 2\beta_{,1}\beta_{,2}\frac{\phi_{,2}}{\phi} - \frac{\omega}{2}\frac{\phi_{,2}^2}{\rho\phi^2} - \frac{\phi_{,2}}{\phi}\left(\frac{\beta_{,2}}{\rho} - \beta_{,1}\frac{\phi_{,2}}{\phi}\right) - \frac{k}{8\pi\phi}\cdot\frac{e^{-2\beta}}{\rho}(1+a^2)(\psi_{,1}^2 - \psi_{,2}^2) - \frac{k}{4\pi}\frac{\phi_{,2}}{\phi^2}e^{-2\beta}(1+a^2)\psi_{,1}\psi_{,2} \quad \ldots(26)$$

$$\alpha_{,2}\left(\frac{1}{\rho^2} + \frac{\phi_{,2}^2}{\phi^2}\right) = -\frac{\phi_{,2}}{\phi}(\beta_{,1}^2 - \beta_{,2}^2) + \frac{2\beta_{,1}\beta_{,2}}{\rho} + \frac{\omega}{2}\frac{\phi_{,2}^3}{\phi^3} + \frac{\phi_{,2}}{\phi}\left(\frac{\beta_{,2}\phi_{,2}}{\phi} + \frac{\beta_{,1}}{\rho}\right) - \frac{k}{4\pi\phi}\frac{e^{-2\beta}}{\rho}(1+a^2)\psi_{,1}\psi_{,2} + \frac{k}{8\pi}\frac{\phi_{,2}}{\phi^2}e^{-2\beta}(1+a^2)(\psi_{,1}^2 - \psi_{,2}^2) \quad \ldots(27)$$

$$\beta_{,11} + \beta_{,22} + \frac{\beta_{,1}}{\rho} = \frac{k}{8\pi\phi}(1+a^2)\, e^{-2\beta}(\psi_{,1}^2 + \psi_{,2}^2) - \frac{\beta_{,2}\phi_{,2}}{\phi} \quad \ldots(28)$$

$$\psi_{,11} + \psi_{,22} + \frac{\psi_{,1}}{\rho} = 2\beta_{,1}\psi_{,1} + 2\beta_{,2}\psi_{,2}. \quad \ldots(29)$$

For convenience we denote eqns. (22) – (25) as set A and eqns. (26) – (29) as set B.

Further substituting

$$\left.\begin{aligned} \beta &= -\log V + \tfrac{1}{2}\log\phi \\ \xi &= -\left(-\frac{2\pi}{k(1+a^2)}\right)^{1/2} 2W \\ \alpha &= 2\mu - 2\log V \\ \omega &= -3/2 \end{aligned}\right\} \quad \ldots(30)$$

in the set A, we get the following field equations (denoted as set C):

$$\mu_{,1}\left(\frac{1}{\rho^2} + \frac{\phi_{,2}^2}{\phi^2}\right) = \frac{1}{\rho}\left(\frac{V_{,1}^2 - V_{,2}^2}{2V^2}\right) + \frac{V_{,1}}{\rho^2 V} - \left(\frac{W_{,1}^2 - W_{,2}^2}{2\rho^3 V^2}\right) + \frac{V_{,2}\phi_{,2}}{\rho\phi V} + \frac{\phi_{,2}}{\phi}\left(\frac{V_{,1}V_{,2}}{V^2} - \frac{W_{,1}W_{,2}}{\rho^2 V^2}\right) \quad \ldots(31)$$

$$\mu_{,2}\left(\frac{1}{\rho^2}+\frac{\phi_{,2}^2}{\phi^2}\right)=-\frac{\phi_{,2}}{\phi}\left(\frac{V_{,1}^2-V_{,2}^2}{2V^2}\right)-\frac{V_{,1}}{\rho V}\frac{\phi_{,2}}{\phi}$$

$$+\frac{\phi_{,2}}{\phi}\left(\frac{W_{,1}^2-W_{,2}^2}{2\rho^2V^2}\right)+\frac{V_{,2}}{\rho^2V}$$

$$+\frac{1}{\rho}\left(\frac{V_{,1}V_{,2}}{V^2}-\frac{W_{,1}W_{,2}}{\rho^2V^2}\right) \qquad ...(32)$$

$$V_{,11}+V_{,22}+\frac{V_{,1}}{\rho}-\frac{(V_{,1}^2+V_{,2}^2)}{V}-\frac{(W_{,1}^2+W_{,2}^2)}{\rho^2V}=-\frac{\phi_{,2}V_{,2}}{\phi} \qquad ...(33)$$

$$W_{,11}+W_{,22}-\frac{W_{,1}}{\rho}-\frac{2(W_{,1}V_{,1}+W_{,1}V_{,2})}{V}=-\frac{\phi_{,1}W_{2}}{\phi}. \qquad ...(34)$$

It can be verified that the set C [viz., eqns. (31) - (34)] corresponds to the B-D vacuum equations for $\omega = 0$ for the generalized stationary symmetric metric

$$ds^2=-e^{\mu}(d\rho^2+dz^2)-\rho^2V\,d\Phi^2+\frac{1}{V}[dt-W\,d\Phi]^2 \qquad ...(35)$$

where μ, V and W are functions of ρ and z only (see Appendix). Hence the theorem.

Theorem — Given $(\alpha, \beta, \xi, \phi_0)$ for $\omega = -\frac{3}{2}$ as B-D Maxwell solutions for the metric (6), one can always construct stationary B-D vacuum solutions (μ, V, W, ϕ) for $\omega = 0$ for the metric (35), as

$$\mu=\frac{\alpha}{2}-\beta+\tfrac{1}{2}\log\phi_0,\; V=e^{-\beta}\phi_0^{1/2}$$

$$W=-\tfrac{1}{2}\xi\left(-\frac{2\pi}{k(1-a^2)}\right)^{-1/2},\; \phi=\phi_0$$

with a change in the sign of the gravitation constant k.

Example: The solutions of the set B [viz., the field eqns. (26) - (29)] have been obtained in our previous paper (Rao and Tiwari 1978) are given as follows:

$$\alpha=-\frac{f^2}{2}\log\left[\frac{1}{\rho^2}+\frac{m^2}{(mz+n)^2}\right]$$

$$+\tfrac{1}{2}(2f^2-f^2m+mf)\log[(mz+n)^2-a^2\rho^2]+s$$

$$\beta=\tfrac{1}{2}\log\left[\frac{k}{8\pi}\,\frac{\{c^2-4d(1+a^2)\}}{1+a^2}\cdot\frac{\rho^{2f}(mz+n)^{2f/m}}{\{\rho^{2f}(mz+n)^{(2f+m)/m}-1\}^2}\right]$$

$$\psi=-\frac{c}{2(1+a^2)}-\frac{\sqrt{c^2-4d(1+a^2)}}{2(1+a^2)}\cdot\frac{\rho^{2f}(mz+n)^{(2f+m)/m}+1}{\rho^{2f}(mz+n)^{(2f+m)/m}-1}$$

$$\phi=mz+n.$$

Using the theorem stated above, we get the solutions for the stationary B-D vacuum fields (for $\omega = 0$) for the metric (35) as

$$V = \left[\frac{k}{8\pi}\,\frac{\{c^2 - 4d(1 + a^2)\}}{(1 + a^2)}\right]^{-1/2} \left[\frac{\rho^{2f}(mz + n)^{(2f+m)/m} - 1}{\rho^f(mz + n)^{(2f-m)/2m}}\right]$$

$$W = \left\{ - \frac{2\pi}{k(1 + a^2)}\right\}^{1/2}$$

$$\times \left[\frac{8\pi f\left(\frac{m^2z^2}{2} + nz\right)}{k\sqrt{c^2 - 4d(1 + a^2)}} + \frac{2\pi(2f + m)\,\rho^2}{k\sqrt{c^2 - 4d(1 + a^2)}} \right]$$

$$\mu = - \frac{f^2}{4} \log\left[\frac{1}{\rho^2} + \frac{m^2}{(mz + n)^2}\right]$$

$$+ \tfrac{1}{4}(2f^2 - f^2m + mf) \log \left[(mz + n)^2 - a^2\rho^2\right]$$

$$+ \log\left[\frac{\rho^{2f}(mz + n)^{(2f+m)/m} - 1}{\rho^f(mz + n)^{(2f-m)/2m}}\right] + s'$$

$$\phi = mz + n$$

where $$s' = \frac{s}{4} - \tfrac{1}{2} \log \left[\frac{k}{8\pi}\,\frac{\{c^2 - 4d(1 + a^2)\}}{(1 + a^2)}\right].$$

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References

Perje's, Z. (1968). A method for constructing certain axially-symmetric Einstein-Maxwell fields. *Il Nuovo Cim.*, **45B**, 600.

Rao, P. P., and Tiwari, R. N. (1978). A class of static Weyl solutions for the Brans-Dicke Maxwell fields. *Acta Physica Hungarica*, **45**, No. 4.

Tauber, G. (1957). The gravitational fields of electric and magnetic dipoles. *Can. J. Phys.*, **35**, 477.

Appendix

The equation $R_{34} = -\frac{1}{\phi}\,\phi_{;34}$ for the metric (35) leads us to $\phi_{,1} = 0$ which when substituted in $\phi_{;s}^{;s} \equiv \phi_{,11} + \phi_{,22} + \frac{\phi_{,1}}{\rho} = 0$, gives $\phi_{,22} = 0$. This on integration yields $\phi = mz + n$ which is same as eqn. (17) for the metric (6).