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# A NOTE ON AN ORTHOGONALITY RELATION FOR STRESS ANALYSIS IN A MULTI-LAYERED LAMINATED DISC

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This note presents an orthogonality relation satisfied by the axial eigenfunctions of the biharmonic equation in cylindrical coordinates, useful for the study of stress analysis in a multi-layered laminated disc.

## INTRODUCTION

In a paper by Rao *at al.* (1976), a biorthogonality relation has been developed for the axial eigenfunctions of the biharmonic Love's strain function. Recently, Fraser (1977) has obtained an orthogonality relation satisfied by these biharmonic functions. Using these biorthogonality/orthogonality relations, solutions of various boundary value problems associated with the axi-symmetric deformation in a thick circular disc can be obtained. However, the existence of an orthogonality relation for a multi-layered circular disc does not follow directly from the analysis of Rao *et al.* (1976) and Fraser (1977). The purpose of this note is to develop a generalised orthogonality relation for a multi-layered laminated circular disc. This orthogonality relation holds good for all physically meaningful combinations of prescribed homogeneous boundary conditions on the plane faces and non-homogeneous self-equilibrating boundary conditions on the circular boundary of the layered disc. Using this orthogonality relation, the stress analysis of problem in a multi-layered laminated disc can be reduced to solving the system of linear simultaneous equations in infinitely many unknowns.

## SOME BASIC RESULTS

Consider an isotropic, homogeneous, elastic circular disc occupying the region $0 \leqslant r \leqslant a$, $-h \leqslant z \leqslant h$ and undergoing axisymmetric deformations. We assume that on the plane faces $z = \pm h$ any admissible homogeneous boundary conditions in stresses and/or displacements can be prescribed in full generality. We seek the expansion of the stresses $\sigma_z$ and $\tau$ and the displacements $u$ and $w$ in the form

$$\begin{bmatrix} u(r, z) \\ \tau(r, z)/2G \end{bmatrix} = \Sigma \, C_n I_1(\lambda_n r/h) \begin{bmatrix} u_n(z) \\ \tau_n(z) \end{bmatrix} \qquad \ldots(1)$$

$$\begin{bmatrix} w(r, z) \\ \sigma_z(r, z)/2G \end{bmatrix} = \Sigma\, C_n I_0(\lambda_n r/h) \begin{bmatrix} w_n(z) \\ \sigma_{zn}(z) \end{bmatrix} \quad ...(2)$$

where the $\lambda_n$'s are eigenvalues and $I_0$, $I_1$ are modified Bessel functions of the zeroth and first order respectively. The constants $G$ and $\mu$ denote the shear modulus and the Poisson's ratio respectively.

Using the analysis outlined in Fraser (1977) and Mary (1972), it can be established that

$$(p_n^2 - p_m^2) \int_{-h}^{+h} \left[\tau_n w_m - u_n \left(\frac{1-\mu}{\mu}\sigma_{zm} - \frac{1}{\mu} w_m\right)\right] dz$$

$$= [p_n(\sigma_{zm} w_n - \sigma_{zn} w_m) + p_m(u_m \tau_n - u_n \tau_m)]_{-h}^{+h} \quad ...(3)$$

where $p_n = \lambda_n/h$. It is important to note that we have expressed the boundary terms appearing on the right-hand side of (3) in terms of the functions $\sigma_{zn}$, $\tau_n$, $u_n$ and $w_n$. It is this step that will ensure the continuity of the stresses and displacements along the common interfaces of a multi-layered disc.

Using the stress-displacement relation (see Timoshenko and Goodier 1970), it can be shown that

$$\sigma_r + 2G\frac{u}{r} = \frac{1-\mu}{\mu}\sigma_z - \frac{2G}{\mu}\frac{\partial w}{\partial z}. \quad ...(4)$$

Defining $t(r, z) = \sigma_r + 2G\dfrac{u}{r}$, we seek the expansions of $t(r, z)$ in the form

$$\frac{t}{2G} = \sum C_n I_0(p_n r)\, t_n(z) \quad ...(5)$$

where

$$t_n = \frac{1-\mu}{\mu}\sigma_{zn} - \frac{1}{\mu} w_n. \quad ...(6)$$

Using (3) and (6), we have

$$(p_n^2 - p_m^2) \int_{-h}^{+h} [\tau_n w_m - u_n t_m]\, dz$$

$$= [p_n(\sigma_{zm} w_n - \sigma_{zn} w_m) + p_m(u_m \tau_n - u_n \tau_m)]_{-h}^{+h}. \quad ...(7)$$

With the following new definitions for $\tau_m$, $u_m$, $w_m$, $t_m$, viz.,

$$[u(r, z), \tau(r, z)] = \sum_n C_n I_1(p_n r)\, [u_n(z), \tau_n(z)]$$

$$[w(r, z), t(r, z)] = \sum_n C_n I_0(p_n r)\,[w_n(z), t_n(z)]$$

it is easy to verify that the relation (7) retains its form. It is important to note that the elastic constants are not appearing on the right-hand side of the relation (7).

When a laminated disc is obtained by bonding two or more homogeneous discs, relation (7) assumes the form

$$\sum_i (p_{in}^2 - p_{im}^2) \int_{-h_i}^{h_i} [\tau_{in} w_{im} - u_{in} t_{im}]\, dz_i$$

$$= \sum_i \left[p_{in}(\sigma_{zim} w_{in} - \sigma_{zin} w_{im}) + p_{im}(u_{im}\tau_{in} - u_{in}\tau_{im})\right]_{-h_i}^{h_i} \qquad \ldots(8)$$

where the summation $i$ is extended over the several homogeneous discs used in forming the laminated disc. The subscript $i$ appearing on $p_n$, $u_n$, $w_n$, $\sigma_{zn}$, $\tau_n$ and $t_n$ denote the corresponding quantities in the $i$th disc. It may be noted that the following combinations of homogeneous boundary conditions may be prescribed on the top and bottom plane faces of the laminated disc:

(i) $\sigma_z = \tau = 0$, (ii) $\sigma_z = u = 0$, (iii) $\tau = w = 0$, (iv) $u = w = 0$.

The right-hand side of (8) vanishes on the top and bottom plane faces of the laminated disc for the above combinations of the boundary conditions. Further, the continuity of the stresses and displacements across the interfaces of the various discs ensure the continuity of the expression appearing within the brackets of the right-hand side of (8). We, therefore, have the orthogonality relation in the form

$$\sum_i \int_{-h_i}^{h_i} [\tau_{in} w_{im} - u_{in} t_{im}]\, dz_i = 0 \quad (m \neq n). \qquad \ldots(9)$$

Let $u_{ib}$, $w_{ib}$, $\tau_{ib}$, $t_{ib}$ $\left(t_{ib} = \sigma_{rib} + 2G_i \dfrac{u_{ib}}{r}\right)$ denote the boundary functions that may be prescribed on $r = a$ of the $i$th disc. We seek the expansions of these functions in the form

$$[u_{ib}, \tau_{ib}] = \sum_n C_n I_1(p_{in} a)\,[u_{in}, \tau_{in}]$$

$$[w_{ib}, t_{ib}] = \sum_n C_n I_0(p_{in} a)\,[w_{in}, t_{in}].$$

Using (9), we get

$$C_n \sum_i I_1(p_{in} a)\, K_{in} = \sum_i \int_{-h_i}^{h_i} [\tau_{ib} w_{in} - u_{ib} t_{in}]\, dz_i \qquad \ldots(10)$$

and

$$C_n \sum_i I_0(p_{in}a)\, K_{in} = \sum_i \int_{-h_i}^{h_i} (w_{ib}\tau_{in} - t_{ib}u_{in}]\, dz_i.$$

The relation (10) shows that a closed form solution exists whenever $u_{ib}$ and $\tau_{ib}$ are prescribed as boundary conditions on $r = a$. Adding the above relations, we get

$$C_n \sum_i [I_0(p_{in}a) + I_1(p_{in}a)]\, K_{in}$$

$$= \sum_i \int_{-h_i}^{h_i} \left[\tau_{ib}w_{in} - u_{ib}t_{in} + w_{ib}\tau_{in} - \left(\sigma_{rib} + 2G_i \frac{u_{ib}}{a}\right) u_{in}\right] dz_i \qquad \ldots(11)$$

where $K_{in} = \int_{-h_i}^{h_i} (\tau_{in}w_{in} - u_{in}t_{in})\, dz_i.$

Using (11), it can be shown that whenever the following combinations of the boundary functions are prescribed on $r = a$, viz.,

(i) $\sigma_{rib}, \tau_{ib}$, (ii) $\sigma_{rib}, w_{ib}$, (iii) $u_{ib}, w_{ib}$

the determination of the constants $C_n$ can be directly reduced to solving a linear algebraic system of equations in infinitely many unknowns. For, if $\sigma_{rib}$ and $\tau_{ib}$ are the prescribed boundary functions, we expand the unknown displacements in the form

$$u_{ib} = \sum_m C_m I_1(p_{im}a)\, u_{im}; \quad w_{ib} = \sum_m C_m I_0(p_{im}a)\, w_{im}.$$

Using (11), we get the system of equations of the type

$$C_n = \sum_m C_m R_{mn} + H_n$$

where $H_n = \sum_i \int_{-h_i}^{h_i} [\tau_{ib}w_{in} - \sigma_{rib}u_{in}]\, dz_i/T_{in}$

and

$$R_{mn} = \sum_i \int_{-h_i}^{h_i} \left[w_{im}\tau_{in}I_0(p_{im}a) - u_{im}\left(t_{in} + \frac{2G_i}{a}u_{in}\right) I_1(p_{im}a)\right] dz_i/T_{in}$$

where $T_{in} = \sum_i [I_0(p_{in}a) + I_1(p_{in}a)]\, K_{in}.$

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