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ON THE ABSOLUTE CESÀRO SUMMABILITY OF ULTRASPHERICAL SERIES

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In this note a theorem is established on the absolute Cesàro summability of ultraspherical series of functions of Lipschitz class.

1. DEFINITIONS

A series Σu_n is said to be summable (C, k) to the sum s, if

$$\lim_{n\to\infty} S_n^k \Rightarrow s$$

where S_n^k is the nth Cesàro mean of order $\alpha > -1$ of the series $\Sigma\, U_n$, and defined as follows:

$$s_n^k = \sum_{\nu=0}^{n} A_\nu^k u_{n-\nu} = \sum_{\nu=0}^{n} A_\nu^{k-1} s_{n-\nu}.$$

The series is said to be summable $| C, k |$ if the sequence $\{s_n^k\}$ is of bounded variation, i.e.

$$\Sigma \mid s_n^k - s_{n-1}^k \mid < \infty.$$

2. INTRODUCTION

Let $f(\theta, \phi)$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$ be defined on a sphere S. The ultraspherical series associated with this function is

$$f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n + \lambda) \iint_S \frac{f(\theta', \phi')\, P_n^{(\lambda)} (\cos \omega) \sin \theta' d\theta' d\phi'}{[\sin^2 \theta' \sin^2 (\phi - \phi')]^{(1-2\lambda)/2}}, \lambda > 0 \quad \text{...(2.1)}$$

where

$$\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$$

and $P_n^{(\lambda)}(x)$ is the well-known ultraspherical polynomial.

We write

$$f(\omega) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)}{\Gamma(\lambda)\,2\pi\,(\sin\omega)^{2\lambda}} \int\limits_{C\omega} \frac{f(\theta',\phi')\sin\theta'\,d\theta'\,d\phi'}{[\sin^2\theta'\,\sin^2(\phi-\phi')]^{(1-2\lambda)/2}}$$

$$F(\omega) = f(\omega)\,(\sin\omega)^{2\lambda-1}. \qquad \ldots(2.2)$$

The integral in (2.2) is taken along the small circle whose centre is (θ, ϕ) on the sphere and whose curvilinear radius is ω.

The series (2.1) now reduces to

$$f(\theta,\phi) \sim \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \sum_{n=0}^{\infty} (n+\lambda) \int_0^{\pi} f(\omega)\,\sin^{2\lambda}\omega\, P_n^{(\lambda)}(\cos\omega)\,d\omega$$

$$\equiv \sum_{n=0}^{\infty} a_n(\theta,\phi). \qquad \ldots(2.3)$$

We prove the following theorem:

Theorem — If $0 < \lambda < \alpha < \frac{1}{2}$ and $0 < n < \frac{1}{2}$

$$F(\omega) \in \text{lip}\,(\eta) \qquad \ldots(2.4)$$

then the series (2.1) is summable $|C, \alpha|$ at a point (θ, ϕ) on the sphere.

3. Proof of the Theorem

Let G_n^{α} the nth Cesàro mean of order α of the sequence $\{na_n(\theta,\phi)\}$; to prove the theorem we have to show only the convergence of $\sum_{n=1}^{\infty} n^{-1}\,|G_n^{\alpha}|$. In the proof of the theorem we choose $\gamma_n = n^{-2/(\eta+2)}$.

Now

$$G_n^{\alpha} = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} F(\omega) \left(\frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}(\nu+\lambda)\,\nu\,P_\nu^{(\lambda)}(\cos\omega)\sin\omega\right) d\omega$$

$$= A \int_0^{\pi} F(\omega) \left(\frac{1}{A_n^{\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}(\nu+\lambda)\,\nu\,P_\nu^{(\lambda)}(\cos\omega)\sin\omega\right) d\omega \qquad \ldots(3.1)$$

where A is a constant independent of n and ν not necessarily the same at each occurrence.

Hence

$$G_n^\alpha = A\,[\int_0^{\gamma_n} + \int_{\gamma_n}^{\pi-n^{-1}} + \int_{\pi-n^{-1}}^{\pi}]. \qquad \ldots(3.2)$$

Therefore, we have

$$\Sigma\, n^{-1}\,|\,G_n^\alpha\,| \leqslant \Sigma\, n^{-1}\{\,|\,I_1\,| + |\,I_2\,| + |\,I_3\,|\,\}$$

where

$$I_1 = A\int_0^{\gamma_n} F(\omega)\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\,\nu\,(\nu+\lambda)\;P_\nu^{(\lambda)}(\cos\omega)\sin\omega\right\}d\omega$$

$$= O\,(n^{2\lambda-1}). \qquad \text{(see Szegö 1959)}$$

$$\sum_{n=1}^{m} n^{-1}\,|\,I_1\,| = O\,[\sum_{n=1}^{m} n^{2\lambda-2}]$$

$$= O(1). \qquad \ldots(3.3)$$

Next

$$I_3 = O\{\int_{\pi-(1/n)}^{\pi}[\sum_{\nu=0}^{n-1} n^{-1}\nu^{2\lambda}\sin\omega + nn^{2\lambda-1}\sin\omega]\,d\omega\}$$

$$\sum_{n=1}^{m} n^{-1}\,|\,I_3\,| = O[\sum_{n=1}^{m} n^{-1}\{\sum_{\nu=0}^{n-1} n^{-2}\nu^{2\lambda} + n^{2\lambda-1}\}]$$

$$= O(\sum_{n=1}^{m} n^{2\lambda-2})$$

$$= O(1). \qquad \text{(see Gupta 1958)} \qquad \ldots(3.4)$$

Finally, we consider I_2. Using the order estimates given by Gupta (1958), we get the following:

$$I_2 = A\int_{\gamma_n}^{\pi-n^{-1}}\sum_{\nu=0}^{n-1}\{R(\Phi(\omega)\,\omega^{-\lambda}e^{i(2n+2\lambda+1)\omega/2})\}\,F(\omega)\,d\omega$$

$$+ \int_{\gamma_n}^{\pi-n^{-1}}\sum_{\nu=0}^{n-1}\{O\,(n^{-2}(\sin\omega)^{-\lambda-1}) + O\,(n^{-1}\nu^{-\lambda}(\sin\omega)^{-\lambda}\,\omega^{-1})$$

$$+ O\,(n^{-1}\nu^{-\lambda-1}(\sin\omega)^{-\lambda-1}) + O\,(n^{-1}\nu^{-\lambda}\omega^{-\lambda}(\sin\omega)^{-1})\}\,F(\omega)\,d\omega$$

$$+ \int_{\gamma_n}^{\pi-n^{-1}} n[R(\psi(\omega)\,\omega^{-\lambda}e^{i(2n+2\lambda+1)\omega/2}]\,F(\omega)\,d\omega$$

$$+ \int_{\gamma_n}^{\pi - n^{-1}} n\{O(n^{\lambda-2}\omega^{-1} (\sin \omega)^{-\lambda})$$

$$+ O(n^{-2} (\sin \omega)^{-\lambda-1}) + O(n^{\lambda-2}\omega^{-\lambda} (\sin \omega)^{-1}\} F(\omega)\, d\omega$$

$$= I_{2\cdot 1} + I_{2\cdot 2} + I_{2\cdot 3} + I_{2\cdot 4} + I_{2\cdot 5} + I_{2\cdot 6} + I_{2\cdot 7} + I_{2\cdot 8} + I_{2\cdot 9}, \text{ say.}$$

Now

$$I_{2\cdot 2} = O\,[(n^{-\eta-2}) \sum_{\nu=0}^{n-1} \{\int_{\gamma_n}^{\pi/2} \omega^{-\lambda-1} d\omega + \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{-\lambda-1}\, d\omega\}]$$

$$\sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 2} \mid = O\,[\sum_{n=1}^{m} n^{-\eta-2+\lambda}]$$

$$= O(1) \qquad \ldots(3.5)$$

$$I_{2\cdot 3} = O\,[(n^{-\eta-1} \sum_{\nu=0}^{n-1} \nu^{-\lambda} \{\int_{\gamma_n}^{\pi/2} \omega^{-1-\lambda} d\omega + \int_{\pi/2}^{\pi-n^{-1}} \omega^{-1} (\sin \omega)^{-\lambda}\, d\omega\}]$$

$$\sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 3} \mid = O\,[\sum_{n=1}^{m} n^{-\eta-1}]$$

$$= O(1) \qquad \ldots(3.6)$$

$$I_{2\cdot 4} = O\,[(n^{-\eta-1}) \sum_{\nu=0}^{n-1} \nu^{-\lambda-1} \{\int_{\gamma_n}^{\pi/2} \omega^{-\lambda-1} d\omega + \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{-\lambda-1}\, d\omega\}]$$

$$\sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 4} \mid = [\sum_{n=1}^{m} n^{-\eta-2}]$$

$$= O(1) \qquad \ldots(3.7)$$

$$I_{2\cdot 5} = O\,[(n^{-\eta-1}) \sum_{\nu=0}^{n-1} \nu^{-\lambda} \{\int_{\gamma_n}^{\pi/2} \omega^{-\lambda-1} d\omega + \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{-1}\, d\omega\}]$$

$$\sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 5} \mid = O\,[\sum_{n=1}^{m} n^{-\eta-1}]$$

$$= O(1). \qquad \ldots(3.8)$$

Again

$$I_{2\cdot 7} = O\,[(n^{-\eta+\lambda-1}) \{\int_{\gamma_n}^{\pi/2} \omega^{-\lambda-1} d\omega + \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{-\lambda}\, d\omega\}]$$

$$\sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 7} \mid = O\,[\sum_{n=1}^{m} n^{-\eta+2\lambda-2}]$$

$$= O(1) \qquad \ldots(3.9)$$

$$I_{2\cdot 8} = O[(n^{-\eta-1}) \{ \int_{\gamma_n}^{\pi/2} \omega^{-\lambda-1}\, d\omega + \int_{\pi/2}^{\pi-n^{-1}} (\sin\omega)^{-\lambda-1}\, d\omega\}]$$

$$+ \sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 8} \mid = O[\sum_{n=1}^{m} n^{\lambda-2-\eta}]$$

$$= O(1) \qquad \ldots(3.10)$$

$$I_{2\cdot 9} = O[(n^{-\eta+\lambda-1}) \{ \int_{\gamma_n}^{\pi/2} \omega^{-\lambda-1} d\omega + \int_{\pi/2}^{\pi-n^{-1}} \omega^{-\lambda} (\sin\omega)^{-1}\, d\omega\}]$$

$$\sum_{n=1}^{m} n^{-1} \mid I_{2\cdot 9} \mid = O[\sum_{n=1}^{m} n^{2\lambda-2-\eta}]$$

$$= O(1). \qquad \ldots(3.11)$$

Also the integral $I_{2\cdot 1}$,

$$\int_{\gamma_n}^{\pi-n^{-1}} \omega^{-\lambda}\, \Phi(\omega)\, F(\omega)\, e^{i(2n+2\lambda+1)\omega/2}\, d\omega$$

$$= \tfrac{1}{2} \{ \int_{\gamma_n}^{\pi-n^{-1}} \omega^{-\lambda}\, \Phi(\omega)\, F(\omega)\, e^{i(2n+2\lambda+1)\omega/2}\, d\omega$$

$$- \int_{\gamma_n-\mu_n}^{\pi-\mu_n-n^{-1}} (\omega+\mu_n)^{-\lambda}\, \Phi(\omega+\mu_n)\, F(\omega+\mu_n)\, e^{i(2n+2\lambda+1)\omega/2}\, d\omega\}$$

and this is less in modulus than

$$\tfrac{1}{2} [\int_{\gamma_n-\mu_n}^{\gamma_n} \mid F(\omega+\mu_n)\, \Phi(\omega+\mu_n)\, (\omega+\mu_n)^{-\lambda} \mid d\omega$$

$$+ \int_{\pi-n^{-1}-\mu_n}^{\pi-n^{-1}} \mid F(\omega)\, \Phi(\omega)\, \omega^{-\lambda} \mid d\omega$$

$$+ \int_{\gamma_n}^{\pi-n^{-1}-\mu_n} \mid F(\omega+\mu_n) - F(\omega) \mid \; \mid \Phi(\omega+\mu_n) \mid (\omega+\mu_n)^{-\lambda}\, d\omega$$

$$+ \int_{\gamma_n}^{\pi-n^{-1}-\mu_n} \mid \Phi(\omega+\mu_n) - \Phi(\omega) \mid \; \mid F(\omega) \mid (\omega+\mu_n)^{-\lambda}\, d\omega$$

$$+ \int_{\gamma_n}^{\pi-\mu_n-n^{-1}} \mid (\omega+\mu_n)^{-\lambda} - \omega^{-\lambda} \mid \; \mid F(\omega) \mid \; \mid \Phi(\omega) \mid d\omega]$$

$$= \tfrac{1}{2} [J_1 + J_2 + J_3 + J_4 + J_5], \text{ say.}$$

Now

$$J_1 = O(\mu_n\, n^{\lambda-\alpha}\, \gamma_n^{1-\alpha}\, \gamma_n^{-\lambda})$$

$$= O(n^{2\lambda-2}) \qquad \ldots(3.12)$$

$$J_2 = O(\mu_n\, n^{\lambda-\alpha})$$
$$= O(n^{\lambda-\alpha-1}) \quad \ldots(3.13)$$

$$J_3 = O((\mu_n)^{\eta} n^{\lambda-\alpha}\, \gamma_n^{1-\alpha}\, \gamma_n^{-\lambda+1})$$
$$= O(n^{-\eta+2\lambda-2}) \quad \ldots(3.14)$$

$$J_4 = O(n^{\lambda-2} \log n \int_{\gamma_n}^{\pi} \omega^{-1}\omega^{\eta}\omega^{-\lambda}\, d\omega)$$
$$= O(n^{\lambda-2} \log n) \quad \ldots(3.15)$$

$$J_5 = O(\mu_n \int_{\gamma_n}^{\pi} \omega^{-\lambda-1} n^{\lambda-\alpha}\omega^{1-\alpha}\, d\omega)$$
$$= O(n^{2\lambda-2}). \quad \ldots(3.16)$$

Hence

$$\sum_{n=1}^{m} n^{-1} \,|\, I_{2\cdot 1} \,| \leqslant \sum_{n=1}^{m} n^{-1} \sum_{\nu=0}^{n-1} \{J_1 + J_2 + J_3 + J_4 + J_5\}$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5, \text{ say.} \quad \ldots(3.17)$$

Now

$$\Sigma_1 = O\{\sum_{n=1}^{m} n^{-1} \sum_{\nu=0}^{n-1} n^{2\lambda-2}\}$$
$$= O(1). \quad \ldots(3.18)$$

Σ_2, Σ_3, Σ_4, Σ_5 can be disposed of in the same way as Σ_1; therefore, we get

$$I_{2\cdot 1} = O(1). \quad \ldots(3.19)$$

$I_{2\cdot 6}$ can be disposed of as $I_{2\cdot 1}$.

Thus the theorem is completely established.

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