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SOME SYMMETRIC GROUPS OF SMALL DEGREES AS AUTOMORPHISM GROUPS OF COMPACT RIEMANN SURFACES

S. C. KALITA

Department of Mathematics, Gauhati University, Gauhati 781014

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In this paper we prove that the symmetric groups S_5, S_{11}, S_{15}, S_{16} occur as automorphism groups of compact Riemann surfaces.

1. INTRODUCTION

Let S denote any compact Riemann surface of genus $g \geqslant 2$. It was proved by Hurwitz (1892) that the set of all the biholomorphic self-transformations, usually called the 'automorphisms', of S onto itself forms a finite group $A(S)$ such that $|A(S)| \leqslant 84(g-1)$. $A(S)$ is called the 'group of automorphisms' of S and any subgroup of $A(S)$ is called an 'automorphism group' of S. A finite group of order $84(g-1)$ representable as an automorphism group of S is usually referred to as a 'maximal automorphism group' or a 'Hurwitz group' of genus g. The smallest value of g for which S admits a Hurwitz group, i.e. a group of $84(g-1)$ automorphisms, is known (see Hurwitz 1892) to be 3, and the corresponding Hurwitz group is the simple group of order 168. We shall see in §2 that Hurwitz groups are homomorphic images of the triangle group (2, 3, 7) and so these groups are perfect. Since no non-trivial symmetric group is perfect, a Hurwitz group cannot be a symmetric group.

In §2 we show that the second and the third maximal orders that an automorphism group of S can attain are $48(g-1)$ and $40(g-1)$ respectively. A finite group will be called an 'M_2-group' or an 'M_3-group' according as it is representable as a group of $48(g-1)$ or $40(g-1)$ automorphisms of S, and an M_2-group and an M_3-group will sometimes be referred to as a 'second maximal' and a 'third maximal' automorphism groups respectively. We shall also see in §2 that an M_2-group and an M_3-group are homomorphic images of the triangle groups (2, 3, 8) and (2, 4, 5) respectively. Thus M_2-groups and M_3-groups are not necessarily perfect indicating the possibility that some of these may be representable as symmetric groups. It was proved by Chetiya (1971) that no symmetric group of degree $\leqslant 17$ can be an M_2-group. Moreover we know that no symmetric group is a Hurwitz group. It is therefore worthwhile to examine if some symmetric groups of small degrees occur as M_3-groups. We discover that the symmetric groups S_n do occur as M_3-groups for $n = 5, 11, 15, 16$.

2. Preliminaries

Let Γ be a Fuchsian group acting on the upper half plane D, and let γ be the genus of the orbit-space D/Γ. If $m_1, m_2, \ldots, m_r$ are the periods of Γ, then Γ is defined by the generators $x_1, x_2, \ldots, x_r, a_1, b_1, \ldots, a_\gamma, b_\gamma$ and the relations

$$x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = x_1 x_2 \ldots x_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1.$$

Such a Fuchsian group will be denoted by $(\gamma; m_1, m_2, \ldots, m_r)$. If $\gamma = 0$ we shall simply use the symbol $(m_1, m_2, \ldots, m_r)$. A subgroup of Γ without elements of finite order except identity is called a 'surface-group'. If F is any fundamental region for Γ, the non-Euclidean area $\triangle(\Gamma)$ of F is given by

$$\triangle(\Gamma) = 2\pi \left\{2\gamma - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right\}.$$

In particular, if K is a surface group with genus g for its orbit-space, then

$$\triangle(K) = 4\pi(g - 1).$$

Moreover if Γ_1 is a subgroup of Γ of finite index, then

$$[\Gamma : \Gamma_1] = \frac{\triangle(\Gamma_1)}{\triangle(\Gamma)}. \qquad \ldots(2.1)$$

Now let S be a compact Riemann surface of genus g, and G an automorphism group of S. Then there exist a Fuchsian group $\Gamma = (\gamma; m_1, m_2, \ldots, m_r)$ and a normal surface group K of Γ such that

$$S = D/K, \; G \cong \Gamma/K \qquad \ldots(2.2)$$

$$|G| = \frac{\triangle(K)}{\triangle(\Gamma)} = \frac{4\pi(g-1)}{2\pi\left\{2(\gamma - 1) + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right\}} \qquad \ldots(2.3)$$

where g is the genus of D/K, i.e. g is the 'orbit-genus' of K (see Macbeath 1965).

It is known (see Siegel 1945) that

$$\triangle(\Gamma) = 2\pi \left\{2(\gamma - 1) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right\} \geqslant \frac{\pi}{21}.$$

Therefore from (2.3) we see that for a fixed g, $|G|$ has the maximum value when $\triangle(\Gamma)$ has the minimum possible positive value. By simple arithmetical calculations one can show that the minimum positive value of $\triangle(\Gamma)$ is attained when $\gamma = 0$, $m_1 = 2, m_2 = 3, m_3 = 7$ and the corresponding maximum value of $|G|$ is $84(g - 1)$

i.e. if G is a Hurwitz group, then $\Gamma = (2, 3, 7)$. From (2.2) we then see that there exists a homomorphism ϕ from the triangle group $\Gamma = (2, 3, 7)$ onto the Hurwitz group G such that Ker ϕ is a surface group of orbit-genus g. Similarly for a fixed g, $|G|$ has the second maximum or the third maximum possible positive value according as $\triangle(\Gamma)$ has the second minimum or the third minimum possible positive value. Again by simple calculations it can be shown that the second minimum or the third minimum positive value of $\triangle(\Gamma)$ is attained when $\gamma = 0$, $m_1 = 2$, $m_2 = 3$, $m_3 = 8$ or $\gamma = 0$, $m_1 = 2$, $m_2 = 4$, $m_3 = 5$ and the corresponding second and third maximum values of $|G|$ are $48(g-1)$ and $40(g-1)$ respectively. Moreover from (2.2) we see that G is a second maximal or a third maximal automorphism group of S according as there exists a homomorphism ϕ of $\Gamma = (2, 3, 8)$ or $\Gamma = (2, 4, 5)$ onto G such that Ker ϕ is a surface group of orbit-genus g.

Theorem 2.1 — A non-trivial finite group G is an M_3-group (i.e. a third maximal automorphism group) if and only if G is generated by two elements a and b satisfying the relations $a^2 = b^4 = (ab)^5 = 1$.

PROOF : If G is an M_3-group, then there is a homomorphism ϕ of $\Gamma = (2, 4, 5)$ onto G. $\Gamma = (2, 4, 5)$ is generated by the elements x_1, x_2, x_3 satisfying the relations $x_1^2 = x_2^4 = x_3^5 = x_1x_2x_3 = 1$. It is easy to see that Γ can also be generated by two elements u and v satisfying $u^2 = v^4 = (uv)^5 = 1$. Then G must be generated by $\phi(u)$ and $\phi(v)$. Now G has order $40(g-1)$, and so neither of the orders of u, v and uv is depressed by ϕ, i.e. we must have $(\phi(u))^2 = (\phi(v))^4 = (\phi(u)\,\phi(v))^5 = 1$. Therefore, G is generated by two elements a and b satisfying $a^2 = b^4 = (ab)^5 = 1$. Conversely, let G be a finite group with generators a and b satisfying

$$a^2 = b^4 = (ab)^5 = 1$$

and Γ the triangle group (2, 4, 5) defined by $u^2 = v^4 = (uv)^5 = 1$. Let $\phi : \Gamma \to G$ be the natural homomorphism which maps u onto a and v onto b, and let K be the kernel of ϕ. Then we have to prove that K has no element of finite order. We know that every elliptic element of Γ is conjugate to a power of one of u, v and uv. If K contained an element of order 2, say, it would then contain either u or v^2, and hence $a = \phi(u) = 1$ and $b^2 = \phi(v)^2 = 1$ and in either case we get a contradiction. So K does not contain any element of order 2. Elements of orders 4 and 5 are similarly disposed of. Thus K is a surface group of genus g, say. Then G is an M_3-group of order $40(g-1)$. This proves the theorem.

Let G be a finite permutation group which acts transitively on N points and which is a homomorphic image of the triangle group (l, m, n). Let G be generated by x, y, z satisfying $x^l = y^m = z^n = xyz = 1$. Let the permutation x have x_i i-cycles ($i = 1, 2, \ldots, l-1$), y have y_j j-cycles ($j = 1, 2, \ldots, m-1$) and z have z_k k-cycles ($k = 1, 2, \ldots, n-1$). Then it is known (see Singerman 1970) that there exists an integer $g \geqslant 0$ such that

$$2g-2+\sum_{i=1}^{l-1} x_i\left(1-\frac{i}{l}\right)+\sum_{j=1}^{m-1} y_j\left(1-\frac{j}{m}\right)$$
$$+\sum_{k=1}^{n-1} z_k\left(1-\frac{k}{n}\right)=N\left(1-\frac{1}{l}-\frac{1}{m}-\frac{1}{n}\right). \qquad \text{...(2.4)}$$

Since an M_3-group is a homomorphic image of (2, 4, 5) we put $l = 2$, $m = 4$, $n = 5$. Then we easily see that $x_i = 0$, $i \neq 1$; $y_j = 0$, $j \neq 1, 2$; $z_k = 0$, $k \neq 1$. Therefore formula (2.4) becomes

$$10x_1 + 15y_1 + 10y_2 + 16z_1 = N + 40(1-g). \qquad \text{...(2.5)}$$

The values of N are restricted in this paper to be less than or equal to 16. We then have $g = 0$ or $g = 1$, for otherwise the right-hand side of (2.5) becomes negative. We thus ultimately get the following formulae for our purpose:

$$10x_1 + 15y_1 + 10y_2 + 16z_1 = N + 40 \qquad \text{...(2.6)}$$

$$10x_1 + 15y_1 + 10y_2 + 16z_1 = N. \qquad \text{...(2.7)}$$

Now for a fixed N the solutions of (2.6) and (2.7) give the possible cycle-structures of the generators of M_3-groups of degree N. We shall say that an M_3-group is of type $\{N/x_1; y_1, y_2; z_1\}$, if N = degree of G, x_1 = number of fixed points of x, y_1 = number of fixed points of y, y_2 = number of 2-cycles of y, z_1 = number of fixed points of z, where x, y, z are the generators of G satisfying $x^2 = y^4 = z^5 = xyz = 1$. Obviously $N \geqslant 5$. Our object in this paper is to show that S_5, S_{11}, S_{15}, S_{16} are M_3-groups, and so solving (2.6) and (2.7) for $N = 5, 11, 15, 16$ only we get the following types of M_3-group:

(i) $\{5/3; 1, 0; 0\}$, (ii) $\{11/1; 1, 1; 1\}$, (iii) $\{15/1; 1, 3; 0\}$,
(iv) $\{15/1; 3, 0; 0\}$, (v) $\{15/3; 1, 1; 0\}$, (vi) $\{16/0; 0, 4; 1\}$,
(vii) $\{16/0; 2, 1; 1\}$, (viii) $\{16/2; 0, 2; 1\}$, (ix) $\{16/4; 0, 0; 1\}$,
(x) $\{16/0; 0, 0; 1\}$.

We shall show that there exists a symmetric group of each of the types (i), (ii), (iv) and (viii). For this we first find the generators of such a group of a given degree and then show that it is the symmetric group of that degree. The generators are obtained by using the 'diagram technique' which we illustrate for the first possibility $\{5/3; 1, 0; 0\}$. In this case x has three fixed points, y has one fixed point, y has no 2-cycles, z has no fixed points. We draw a pentagonal figure [see Fig. 1] whose vertices (taken in a certain order, say anticlockwise) denote the 5-cycle of z. A fixed point

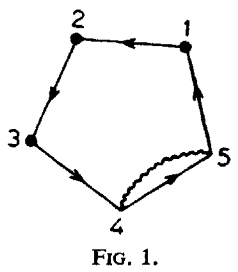

FIG. 1.

of x is denoted by a heavy dot in the figure and a 2-cycle in x is denoted by two vertices joined by a curly line. Two fixed points of x may occur at consecutive points (vertices), one point apart, two points apart, and so on. We start at a point, traverse along a curly line (if it is a fixed point, we do not move at this instance, but still we count it as a move) and then go along an edge in the preassigned direction and stop at the end point. Now if we see that we have arrived at the starting point, then that point is a fixed point of $xz = \alpha$, say. If it is not so we repeat this process once more and see if we have arrived back at the starting point and if we have, then the starting point and the point we stopped at represent in that order the points of a 2-cycle of α. If we have not arrived back at the starting point we are to repeat the process twice more and see if we have arrived back at the starting point, and if we have, then the points we stopped at will represent in that order, the points of a 4-cycle of α. In our present case $x = (1)(2)(3)(4, 5)$, $\alpha = xz = (1, 2, 3, 4)(5)$ and $z = (1, 2, 3, 4, 5)$. If $\{x, y, z\}$ denotes the group generated by x, y, z, then x and α are sufficient to generate $\{x, y, z\}$, for $z = x\alpha$ and $y = x\alpha^{-1}x$ where $x^2 = \alpha^4 = (x\alpha)^5 = 1$. From $y = x\alpha^{-1}x$, it follows that the cycle structure of y is same as that of α. If the cycle structure of α obtained from the figure is not according to our requirement, then the placing of the fixed points of x in the diagram and joining the remaining vertices into pairs by curly lines need re-adjustment. For $N > 5$, the generator z may have fixed points which are denoted by cross marks lying outside the pentagon(s) and each of these points is joined with a vertex of the pentagon(s) by a curly line. The rest of the procedure is then similar to that explained above.

We now list below the generators x and $\alpha = xz$ of the M_3-groups of types (ii), (iv) and (viii) obtained with the help of Figs. 2, 3 and 4 respectively.

Type (ii) $\{11/1; 1, 1; 1\}$

$$x = (2)(1, 4)(3, 6)(5, 7)(8, 11)(9, 10)$$

$$\alpha = xz = (1, 5, 8, 7)(2, 3, 6, 4)(9, 11)(10)$$

Type (iv) $\{15/1; 3, 0; 0\}$

$$x = (1)(2, 6)(3, 4)(5, 11)(7, 15)(8, 9)(10, 12)(13, 14)$$

$$\alpha = xz = (1, 2, 7, 11)(3, 5, 12, 6)(4)(8, 10, 13, 15)(9)(14)$$

Type (viii) $\{16/2; 0, 2; 1\}$

$$x = (1)(12)(2, 9)(3, 7)(4, 6)(5, 15)(8, 11)(10, 14)(13, 16)$$

$$\alpha = xz = (4, 7)(1, 2, 10, 15)(3, 8, 11, 9)(5, 16, 14, 6)(12, 13)$$

In the following section x and α will be denoted for the sake of uniformity by a and b respectively.

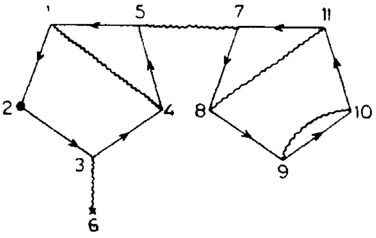

FIG. 2.

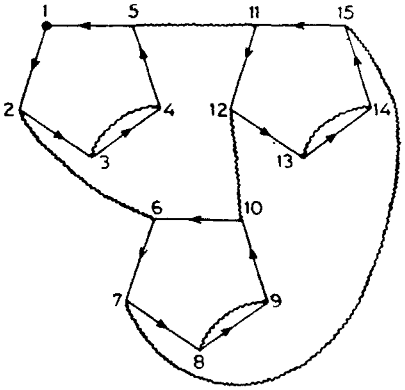

FIG. 3.

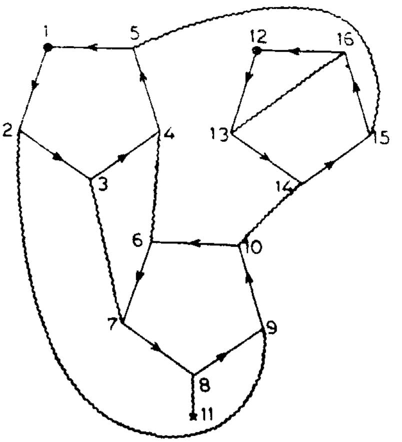

FIG. 4.

3. S_5, S_{11}, S_{15}, S_{16} AS M_3-GROUPS

We now prove that $S_5, S_{11}, S_{15}, S_{16}$ are all M_3-groups. We state below two well-known results (see Hall 1958) for future reference.

(A) The order of a k-ply transitive group of degree n is divisible by

$$n(n-1) \ldots (n-(k-1)).$$

(B) Let the integer $n = kp + r$, where p is a prime and $p > k$, $r > k$ except $k = 1, r = 2$. Then a group of degree n cannot be as much as $(r+1)$-fold transitive unless it is S_n or A_n, the alternating group on n symbols.

Theorem 3.1 — S_5 is an M_3-group.

PROOF : We have already seen that $G = \{a, b\}$, where

$$\left.\begin{aligned} a &= (1)(2)(3)(4, 5) \\ b &= (1, 2, 3, 4)(5) \end{aligned}\right\} \qquad \ldots(3.1.1)$$

and so $a^2 = b^4 = (ab)^5 = 1$, is an M_3-group of type $\{5/3;\ 1, 0;\ 0\}$.

We now prove that $G = \{a, b\}$ is the symmetric group S_5. We immediately get $ab = (1, 2, 3, 4, 5)$. We now have the following permutations in G:

$$b^2 = (1, 3)(2, 4)(5)$$

$$b^{-1} = (1, 4, 3, 2)(5)$$

$$(ab)^2 = (1, 3, 5, 2, 4)$$

$$ab^2 = (1, 3)(2, 4, 5)$$

$$(ab^2)^2 = (1)(3)(2, 5, 4) \qquad \ldots(3.1.2)$$

$$b^{-1}(ab)^2 = (1)(2, 3, 4, 5). \qquad \ldots(3.1.3)$$

Now (3.1.3) $\Rightarrow G_1$ is transitive

(3.1.2) $\Rightarrow G_{1,3}$ is transitive

(3.1.1) $\Rightarrow G_{1,3,2}$ is transitive.

Thus G is 4-transitive. By (A) we know that $|G|$ is divisible by $5.4.3.2$ i.e., by 120. Moreover G is a subgroup of S_5 and the order of S_5 is 120. Hence $|G| \leqslant 120$. But since $|G|$ is divisible by 120, $|G| = 120$ showing that $G = S_5$.

Theorem 3.2 — S_{11} is an M_3-group.

PROOF : We have already obtained that $G = \{a, b\}$, where

$$a = (2)(1, 4)(3, 6)(5, 7)(8, 11)(9, 10)$$

$$b = (10)(1, 5, 8, 7)(2, 3, 6, 4)(9, 11)$$

and so $a^2 = b^4 = (ab)^5 = 1$, is an M_3-group of type $\{11/1;\ 1, 1;\ 1\}$ and it is a subgroup of S_{11}. We now prove that $G = \{a, b\}$ is the symmetric group S_{11}. We see that $ab = (6)(1, 2, 3, 4, 5)\ (7, 8, 9, 10, 11)$. We now have the following permutations in G:

$$b^2 = (9)(10)(11)(1, 8)(5, 7)(2, 6)(3, 4) \qquad \ldots(3.2.1)$$

$$a(ab)^3 = (1, 2, 5, 10, 7, 3, 6)(8, 9)(4)(11) \qquad \ldots(3.2.2)$$

$$\{a(ab)^3\}^2 = (1, 5, 7, 6, 2, 10, 3)(4)(8)(9)(11) \qquad \ldots(3.2.3)$$

$$ab^2 = (1, 3, 2, 6, 4, 8, 11)(9, 10)(5)(7)$$

$$(ab^2)^{-1}(ab)^{-1} = (1, 10, 8, 3, 5, 4, 6)\,(7, 11)\,(2)\,(9)$$

$$u = \{(ab^2)^{-1}(ab)^{-1}\}^2 = (1, 8, 5, 6, 10, 3, 4)\,(2)\,(7)\,(9)\,(11)$$

$$x = a^{-1}b^{-1}ab = (1, 6, 3, 4, 8, 10, 7, 2, 5, 9, 11)$$

$$y = bab^2a = (1, 7, 6, 11, 9, 4, 3)\,(5, 8)\,(2)\,(10)$$

$$\alpha = x^{-2}y^{-1} = (1, 11, 8, 4, 7, 5)\,(2, 10, 9)\,(3)\,(6)$$

$$ab\alpha = (1, 10, 8, 2, 3, 7, 4)\,(5, 11)\,(6)\,(9)$$

$$u(ab\alpha)^4 = (1, 4, 3, 2)\,(5, 6, 7, 8)\,(9)\,(10)\,(11) \qquad \ldots(3.2.4)$$

$$u(ab\alpha)^4 b^2 = (1, 3, 6, 5, 2, 8, 7)\,(4)\,(9)\,(10)\,(11). \qquad \ldots(3.2.5)$$

Now (3.2.1) and (3.2.2) $\Rightarrow G_{11}$ is transitive

(3.2.1) and (3.2.3) $\Rightarrow G_{11,9}$ is transitive

(3.2.1) and (3.2.4) $\Rightarrow G_{11,9,10}$ is transitive

(3.2.5) $\Rightarrow G_{11,9,10,4}$ is transitive.

Thus G is 5-transitive. We have $11 = 1.7 + 4$ and so by (B) G is either S_{11} or A_{11}. But $G \neq A_{11}$, since both the generators of G are odd. Hence $G = S_{11}$.

Theorem 3.3 — S_{15} is an M_3-group.

PROOF : We have seen that $G = \{a, b\}$, when

$$a = (1)\,(2, 6)\,(3, 4)\,(5, 11)\,(7, 15)\,(8, 9)\,(10, 12)\,(13, 14)$$

$$b = (1, 2, 7, 11)\,(3, 5, 12, 6)\,(8, 10, 13, 15)\,(4)\,(9)\,(14)$$

and so $a^2 = b^4 = (ab)^5 = 1$, is an M_3-group of type $\{15/1; 3, 0; 0\}$ and it is a subgroup of S_{15}. We now prove that G is the symmetric group S_{15}. We have the following permutations in G:

$$x = a^{-1}b^{-1}ab = (1, 12, 9, 11, 4, 7, 14, 6, 2, 13, 15, 3, 5, 8, 10)$$

$$x^2 = (1, 9, 4, 14, 2, 15, 5, 10, 12, 11, 7, 6, 13, 3, 8)$$

$$x^3 = (1, 11, 14, 13, 5)\,(12, 4, 6, 15, 8)\,(9, 7, 2, 3, 10)$$

$$y = bab^2a = (1, 11, 15, 8, 4, 10, 13)\,(2, 12, 5, 7)\,(3, 6)\,(9, 14)$$

$$y^3 = (1, 8, 13, 15, 10, 11, 4)\,(2, 7, 5, 12)\,(3, 6)\,(9, 14)$$

$$y^7 = (1)\,(4)\,(8)\,(10)\,(11)\,(13)\,(15)\,(3, 6)\,(9, 14)\,(2, 7, 5, 12) \qquad \ldots(3.3.1)$$

$$x^2y = (1, 14, 12, 15, 7, 3, 4, 9, 10, 5, 13, 6)\,(2, 8, 11)$$

$$(x^2y)^3 = (2)\,(8)\,(11)\,(1, 15, 4, 5)\,(14, 7, 9, 13)\,(12, 3, 10, 6) \qquad \ldots(3.3.2)$$

$$x^2y^3 = (1, 14, 7, 3, 13, 6, 15, 12, 4, 9)(2, 10)(5, 11)(8) \quad ...(3.3.3)$$

$$(x^2y^3)^2 = (1, 7, 13, 15, 4)(14, 3, 6, 12, 9)(2)(5)(8)(10)(11) \quad ...(3.3.4)$$

$$x^3y = (1, 15, 4, 3, 13, 7, 12, 10, 14)(2, 6, 8, 5, 11, 9)$$

$$(x^3y)^6 = (2)(5)(6)(8)(9)(11)(1, 12, 3)(15, 10, 13)(4, 14, 7). \quad ...(3.3.5)$$

Now (3.3.2) and (3.3.3) $\Rightarrow G_8$ is transitive

(3.3.1) and (3.3.2) $\Rightarrow G_{8,11}$ is transitive

(3.3.2) and (3.3.4) $\Rightarrow G_{8,11,2}$ is transitive

(3.3.4) and (3.3.5) $\Rightarrow G_{8,11,2,5}$ is transitive.

Thus G is 5-transitive. Now, since $15 = 1.11 + 4$ by (B) G is either S_{15} or A_{15}. But both the generators of G are odd. Therefore G is S_{15}.

Theorem 3.4 — S_{16} is an M_3-group.

PROOF : We have already seen that $G = \{a, b\}$, where

$$a = (1)(12)(2, 9)(3, 7)(4, 6)(5, 15)(8, 11)(10, 14)(13, 16)$$

$$b = (4, 7)(1, 2, 10, 15)(3, 8, 11, 9)(5, 16, 14, 6)(12, 13)$$

so that $a^2 = b^4 = (ab)^5 = 1$, is an M_3-group of type $\{16/2; 0, 2; 1\}$ and it is a subgroup of S_{16}. We now prove that $G = \{a, b\}$ is the symmetric group S_{16}. We have the following permutations in G:

$$b^2 = (4)(7)(12)(13)(1, 10)(2, 15)(3, 11)(8, 9)(5, 14)(6, 16) \quad ...(3.4.1)$$

$$ab^2 = (1, 10, 5, 2, 8, 3, 7, 11, 9, 15, 14)(4, 16, 13, 6)(12) \quad ...(3.4.2)$$

$$(ab^2)^4 = (1, 8, 9, 10, 3, 15, 5, 7, 14, 2, 11)(4)(6)(12)(13)(16) \quad ...(3.4.3)$$

$$(ab)^3 = (11)(1, 4, 2, 5, 3)(6, 9, 7, 10, 8)(12, 15, 13, 16, 14)$$

$$a(ab)^3 = (1, 4, 9, 5, 13, 14, 8, 11, 6, 2, 7)(3, 10, 12, 15)(16)$$

$$\alpha = \{a(ab)^3\}^4 = (1, 13, 6, 4, 14, 2, 9, 8, 7, 5, 11)(3)(10)(12)(15)(16) \quad ...(3.4.4)$$

$$(ab)^{-1} = (11)(1, 5, 4, 3, 2)(6, 10, 9, 8, 7)(12, 16, 15, 14, 13)$$

$$a(ab)^{-1} = (1, 5, 14, 9)(2, 8, 11, 7)(3, 6)(4, 10, 13, 15)(12, 16)$$

$$\beta = \{a(ab)^{-1}\}^2 = (1, 14)(5, 9)(2, 11)(8, 7)(4, 13)(10, 15)(3)(6)(12)(16)$$

$$\alpha^2\beta = (1, 6)(2, 7)(4, 11)(5, 14)(8, 9)(10, 15)(3)(12)(13)(16). \quad (3.4.5)$$

Now (3.4.2) and (3.4.4) $\Rightarrow G_{12}$ is transitive

(3.4.1), (3.4.3) and (3.4.5) $\Rightarrow G_{12,13}$ is transitive

(3.4.3) and (3.4.5) $\Rightarrow G_{12,13,16}$ is transitive.

Thus G is 4-transitive. By (B) we see that $G = S_{16}$.

4. Some More M_3-groups

As a result of our systematic investigation of M_3-groups of various types of degrees between 5 and 16 we have found, in addition to the symmetric groups mentioned in §3, the following finite groups occurring as M_3-groups:

(1) A_6

(2) G_{160} (a group of order 160 defined by $R^4 = S^5 = (RS)^2 = (R^{-1}S)^4 = 1$)

(3) G_{720} (a group of order 720, the derived group being isomorphic to A_6)

(4) G_{1320} (a group of order 1320, the derived group being isomorphic to L.F. (2, 11))

The detailed proofs of these results will be given in a later paper.

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References

Chetiya, B. P. (1971). Groups of automorphisms of compact Riemann surfaces. Ph.D. Thesis, Birmingham University.

Hall, M. (1958). The Theory of Groups. Macmillan & Co., London.

Hurwitz, A. (1892). Uber algebraische Gebilde mit eindeutigen transformationen in sich. *Math. Ann.*, **41**, 403-42.

Macbeath, A. M. (1965). On a curve of genus 7. *Proc. Lond. math. Soc., Third Series*, **14**, 527-42.

Siegel, C. L. (1945). Some remarks on discontinuous groups. *Ann. Math.*, **46**, 708-18.

Singerman, D. (1970). Subgroups of Fuchsian groups and finite permutation groups. *Bull. Lond. math. Soc.*, **2**, 319-23.