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ON CERTAIN SCHLICHT MAPPINGS

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Let U denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in the unit disc $E = \{z : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. In this paper, the authors prove distortion theorems and coefficient estimates of the subclasses of analytic functions introduced by them; namely starlike, convex and close-to-convex w.r.t. n-ply symmetrical point.

1. INTRODUCTION

Definition 1 — A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to the class U belongs to the class of functions starlike w.r.t. n-ply symmetrical points of order β, $0 \leqslant \beta < 1$, if it satisfies the condition.

$$\text{Re}\left\{\frac{zf'(z)}{f_n(z)}\right\} > \beta \; |z| < 1 \qquad \ldots(1.1)$$

where $f_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{n-k} f(z\omega^k)$, ω being the nth root of unity. We shall denote this class with the symbol $S_s^{*,n,\beta}$.

Definition 2 — A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to the class U is said to be convex w.r.t. n-ply symmetrical points of order β, $0 \leqslant \beta < 1$, if it satisfies the condition

$$\text{Re}\left[\frac{\{zf'(z)\}'}{f_n'(z)}\right] > \beta, \; |z| < 1. \qquad \ldots(1.2)$$

Let the class of such functions be symbolized by the notation $C_s^{n,\beta}$.

Definition 3 — A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class of functions close-to-convex w.r.t. n-ply symmetrical points of order β, $0 \leqslant \beta < 1$, corresponding to the function $h(z)$ in $C_s^{n,0}$ if it satisfies the condition

$$\text{Re}\left[\frac{f'(z)}{h_n'(z)}\right] > \beta \mid z \mid < 1 \qquad \text{...(1.3)}$$

where $$h_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{n-k} h(z\omega^k).$$

Let this class be denoted by the symbol $K_s^{n,\beta}$.

Definition 4 — A function $f(z) = z + \sum_{p=1}^{\infty} a_{np+1} z^{np+1}$ belongs to the class $S^*(\beta, n)$ if it satisfies the following

$$\text{Re}\left[z\frac{f'(z)}{f(z)}\right] > \beta \mid z \mid < 1, 0 \leqslant \beta < 1. \qquad \text{...(1.4)}$$

2. Distortion Theorem

Theorem 2.1 — If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $S_s^{*,n,\beta}$, then

$$\int_0^r \frac{1 - 2(1-\beta)\,\rho + (1-2\beta)\,\rho^2}{(1-\rho^2)\,(1+\rho^n)^{2(1-\beta)/n}}\, d\rho \leqslant \mid f(z) \mid$$

$$\leqslant \int_0^r \frac{1 + 2(1-\beta)\,\rho + (1-2\beta)\,\rho^2}{(1-\rho^2)\,(1-\rho^n)^{2(1-\beta)/n}}\, d\rho. \qquad \text{...(2.1)}$$

Sharpness on the R.H.S. of (2.1) is given by the function

$$g(z) = \int_0^z \frac{1 + 2(1-\beta)\,\rho + (1-2\beta)\,\rho^2}{(1-\rho^2)\,(1-\rho^n)^{2(1-\beta)/n}}\, d\rho. \qquad \text{...(2.2)}$$

Inequality on L.H.S. of (2.1) is also sharp.

Proof : Since $f(z)$ $S_s^{*,n,\beta}$, we have

$$\text{Re}\left[\frac{zf'(z)}{f_n(z)}\right] > \beta, \mid z \mid < 1.$$

In view of a result by Libera and Livingston (1972), we have

$$\frac{1-2(1-\beta)r+(1-2\beta)r^2}{1-r^2} \leqslant \left|\frac{zf'(z)}{f_n(z)}\right|$$

$$\leqslant \frac{1+2(1-\beta)r+(1-2\beta)r^2}{(1-r^2)}. \qquad \ldots(2.3)$$

Since $f_n(z) \in S^*(\beta, n)$, in view of (2.3) we have

$$\int_0^r \frac{1-2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho^2)(1+\rho^n)^{2(1-\beta)/n}}\,d\rho \leqslant |f(z)|$$

$$\leqslant \int_0^r \frac{1+2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho^2)(1-\rho^n)^{2(1-\beta)/n}}\,d\rho.$$

To consider the function (2.2), we have

$$g(z) = \int_0^z \frac{1+2(1-\beta)t+(1-2\beta)t^2}{(1-t^2)(1-t^n)^{2(1-\beta)/n}}\,dt.$$

Replacing z by $\omega z, \omega^2 z, \ldots, \omega^{n-1}z$ in the above expression, we obtain

$$g_n(z) = \int_0^z \frac{1+(1-2\beta)t^n}{(1-t^n)^{1+(2(1-\beta)/n)}}\,dt$$

$$= \frac{z}{(1-z^n)^{2(1-\beta)/n}}. \qquad \ldots(2.4)$$

In view of (2.4), we have

$$\text{Re}\left[\frac{zg'(z)}{g_n(z)}\right] = \text{Re}\left[\frac{1+2(1-\beta)z+(1-2\beta)z^2}{(1-z^2)}\right] > \beta.$$

Therefore the function $g(z) \in S_s^{*,n,\beta}$.

It is not so difficult to prove that the function $g(z)$ gives sharpness on the R.H.S. of inequality (2.2). Similarly we can prove that the L.H.S. of (2.2) is also sharp.

Substituting $n = 2$ in Theorem 2.1, we obtain a result by Singh (1977).

Theorem 2.2 — If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $C_s^{n,\beta}$, then

$$\frac{1}{r}\int_0^r \frac{1-2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho)^2(1+\rho^n)^{2(1-\beta)n}}\,d\rho \leqslant |f'(z)|$$

$$\leqslant \frac{1}{r}\int_0^r \frac{1+2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho^2)(1-\rho^n)^{2(1-\beta)/n}}\,d\rho. \qquad \text{...(2.5)}$$

The result (2.5) is sharp.

PROOF : Since $f(z) \in C_s^{n,\beta}$, this implies $zf'(z) \in S_s^{*,n,\beta}$. Now applying Theorem 2.1 to $zf'(z)$ with $z = re^{i\theta}$, we obtain the relation (2.5).

This completes the proof of theorem.

Taking $n = 2$ in Theorem 2.2, we get a result due to Singh (1977).

Taking $\beta = 0$, $n = 2$; we obtain the following result due to Sakaguchi (1959).

Corollary — If the function $f(z) \in C_s^{2,0}$, then

$$\frac{1}{r}\int_0^r \frac{1-\rho}{(1+\rho)(1+\rho^2)}\,d\rho \leqslant |f'(z)| \leqslant \frac{1}{1-r}. \qquad \text{...(2.6)}$$

Theorem 2.3 — If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $K_s^{n,\beta}$ w.r.t. $g(z)$ in $S_s^{*,n,0}$, then

$$\int_0^r \frac{1-2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho^2)(1+\rho^n)^{(n/2)}}\,d\rho \leqslant |f(z)|$$

$$\leqslant \int_0^r \frac{1+2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho^2)(1-\rho^n)^{n/2}}\,d\rho. \qquad \text{...(2.7)}$$

The sign of equality on the R.H.S. of (2.7) is attained by the function

$$f_1(z) = \int_0^z \frac{1+2(1-\beta)\rho+(1-2\beta)\rho^2}{(1-\rho^2)(1-\rho^n)^{n/2}}\,d\rho$$

belonging to $K_s^{n,\beta}$ w.r.t. $g_1(z) = \frac{z}{(1-z^n)^{n/2}}$ in $S_s^{*,n,0}$. The inequality on the L.H.S. of (2.7) is also sharp.

PROOF : Since the function $f(z) \in K_s^{n,\beta}$ w.r.t. $g(z)$ in $S_s^{*,n,0}$, we have

$$\operatorname{Re}\left[\frac{zf'(z)}{g_n(z)}\right] > \beta, \ |z| < 1.$$

Now in view of a result due to Libera and Livingston (1972), we get

$$\frac{1 - 2(1-\beta)\, r + (1-2\beta)\, r^2}{(1-r^2)} \leqslant \left|\frac{zf'(z)}{g_n(z)}\right|$$

$$\leqslant \frac{1 + 2(1-\beta)\, r + (1-2\beta)\, r^2}{(1-r^2)}. \qquad \ldots(2.8)$$

Again $g(z) \in S_s^{*,n,0}$ implies that $g_n(z) \in S^*(\beta, n)$. Therefore, (2.8) yields

$$\int_0^r \frac{1 - 2(1-\beta)\,\rho + (1-2\beta)\,\rho^2}{(1-\rho^2)(1+\rho^n)^{n/2}}\, d\rho \leqslant |f(z)|$$

$$\leqslant \int_0^r \frac{1 + 2(1-\beta)\,\rho + (1-2\beta)\,\rho^2}{(1-\rho^2)(1-\rho^n)^{n/2}}\, d\rho.$$

Also, we have

$$g_1(z) = \frac{z}{(1-z^n)^{n/2}}.$$

Therefore

$$\operatorname{Re}\left[\frac{zf_1'(z)}{g_{1,n}(z)}\right] = \operatorname{Re}\left[\frac{1 + 2(1-\beta)\, z + (1-2\beta)\, z^2}{1-z^2}\right] > \beta.$$

Moreover,

$$\operatorname{Re}\left[\frac{zg_1'(z)}{g_{1,n}(z)}\right] = \operatorname{Re}\left[\frac{zg_1'(z)}{g_1(z)}\right] > 0.$$

This proves that $f_1(z) \in K_s^{n,\beta}$ w.r.t. $g_1(z)$ in $S_s^{*,n,0}$. It is not difficult to prove sharpness of the result.

Taking $n = 2$, in Theorem 2.3 we obtain a result due to Singh (1977). Taking $\beta = 0$ and $n = 2$, Theorem 2.3 yields a result due to Sakaguchi (1959).

3. COEFFICIENT ESTIMATES

In this section, we shall prove the results concerning coefficient bounds of the functions belonging to the classes $S_s^{*,k,\beta}$, $C_s^{k,\beta}$ and $K_s^{k,\beta}$.

Theorem 3.1 — If the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $S_s^{*,k,\beta}$, then

$$| a_m | \leqslant \frac{1}{m} \left[2(1-\beta) \left\{ \sum_{r=1}^{r=t} \frac{p(p+1) \dots (p+r-1)}{r!} + 1 \right\} \right] \quad \dots(3.1)$$

when $\qquad p = \frac{2}{k}(1-\beta),\ tk + 1 < m \leqslant (t+1)\,k$

and

$$| a_m | \leqslant \frac{1}{m} \left[\frac{p(p+1) \dots (p+t-1)}{t!} + 2(1-\beta) \left\{ \sum_{r=1}^{t-1} \frac{p(p+1) \dots (p+r-1)}{r!} + 1 \right\} \right] \quad \dots(3.2)$$

when $\qquad m = tk + 1.$

The sign of equality being attained for the function

$$f(z) = \int_0^z \frac{1 + (1-2\beta)\,\rho}{(1-\rho)\,(1-\rho^k)^{2(1-\beta)/k}}\, d\rho. \quad \dots(3.3)$$

PROOF : Let

$$P(z) = \frac{(zf'(z)/f_k(z)) - \beta}{1-\beta} = 1 + c_1 z + c_2 z^2 + \dots .$$

Now $P(z)$ is analytic and Re $P(z) > 0$ with $P(0) = 1$ for $| z | < 1$. Therefore by a well known result by Goluzin (1969), we have $| c_n | \leqslant 2$ for $n = 1, 2, \dots$. This is, the coefficients of $P(z)$ are bounded by the coefficients of $\frac{1+z}{1-z}$. Since $f(z) \in S_s^{*,k,\beta}$, this implies $f_k(z) \in S^*(\beta, k)$ and the coefficients of $f_k(z)$ are bounded by the coefficients of

$$H(z) = \frac{z}{(1-z^k)^{2(1-\beta)/k}}.$$

The coefficients of $\frac{1+z}{1-z}$ and $H(z)$ being positive numbers, we conclude that the coefficients of $zf'(z)$ are bounded by the respectively coefficients of

$$h(z) = [1 + (1-\beta)\,(c_1 z + c_2 z^2 + \dots)]\, \frac{z}{(1-z^k)^{2(1-\beta)/k}}.$$

The above argument yields

$$zf'(z) \ll 1 + (1 - \beta)\, 2(z + z^2 + \ldots) \times \left\{ z + pz^{k+1} + \ldots + \frac{(p+1)\ldots(p+m-1)}{m!} z^{mk+1} + \ldots \right\} \quad \ldots(3.4)$$

where $p = 2(1 - \beta)/k$.

$f(z) \ll g(z)$ means, the function $f(z)$ is majorized by $g(z)$ (for definition see Keogh and Miller 1971).

Now comparing coefficients on both the sides of (3.4), we get (3.1) and (3.2).

For the function defined by (3.3), we have

$$\operatorname{Re}\left[\frac{zf'(z)}{f_n(z)}\right] > \beta.$$

Therefore the function defined by (3.3) belongs to $S_s^{*,n,\beta}$.

This completes the proof of the theorem.

Taking $k = 2$, $\beta = 0$, we get a result due to Sakaguchi (1959).

Theorem 3.2 — Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to the class $C_s^{n,\beta}$, then

$$| a_m | \leqslant \frac{1}{m^2}\left[2(1 - \beta)\left\{ \sum_{r=1}^{r=t} \frac{p(p+1)\ldots(p+r-1)}{r!} + 1 \right\}\right], \quad \ldots(3.5)$$

when $tk + 1 < m \leqslant (t + 1)k$,

and

$$| a_m | \leqslant \frac{1}{m^2}\left[\frac{p(p+1)\ldots(p+t-1)}{t!} + 2(1 - \beta)\left\{ \sum_{r=1}^{t-1} \frac{(p+1)\ldots(p+r-1)}{r!} + 1 \right\}\right] \quad \ldots(3.6)$$

when $m = tk + 1$ and $p = 2(1 - \beta)/k$.

The sign of equality being attained by the function $f(z)$ given by

$$zf'(z) = \int_0^z \frac{1 + (1 - 2\beta)\,\rho}{(1 - \rho)\,(1 - \rho^k)^{2(1-\beta)/k}}\, d\rho.$$

Proof: Since $f(z) \in C_s^{n,\beta}$, this implies $zf'(z) \in S_s^{*,n,\beta}$. Theorem 3.1 yields (3.5) and (3.6).

The following theorem can be proved just on the lines of Theorem 3.1 and hence the proof is omitted.

Theorem 3.3 — Let the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to $K_s^{n,\beta}$ w.r.t. $g(z)$ in $S_s^{*,n,0}$, then

$$| a_m | \leqslant \frac{1}{m}\left[2(1 - \beta)\left\{\sum_{r=1}^{t} \frac{p(p+1)\dots(p+r-1)}{r!} + 1\right\}\right] \qquad \dots(3.7)$$

when $tk + 1 < m \leqslant (t+1)\,k$,

and

$$| a_m | \leqslant \frac{1}{m}\left[\frac{p(p+1)\dots(p+t-1)}{t!} + 2(1 - \beta)\left\{\sum_{r=1}^{t-1} \frac{p(p+1)\dots(p+r-1)}{r!} + 1\right\}\right] \qquad \dots(3.8)$$

when $m = tk + 1$ and $p = 2\,(1 - \beta)/k$.

These inequalities are sharp.

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