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SLOW MOTION OF A SPHEROID IN A ROTATING FLUID

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The uniform slow motion of a spheroid along the axis of rotation of an infinite mass of liquid in an inviscid, incompressible fluid is examined. The pressure and velocity field on the spheroid and ultimate velocity distribution for various cases are discussed. The cases of oblate and prolate spheroids are considered separately and the basic difference in the ultimate upstream flow pattern is analytically projected. It is observed that in both cases a liquid cylinder of the same radius as that of spheroid is pushed along with it. In case of oblate spheroid there is a swirling motion about the axis of the cylinder while for the prolate spheroid the liquid is stagnant inside the cylinder. The ultimate motion outside the cylinder is steady in case of oblate and time-dependent for the prolate.

1. INTRODUCTION

When an axisymmetric body moves along its axis of revolution in a rotating fluid the disturbance created by it has a singular character on the circumscribing cylinder and the fluid inside the cylinder is pushed along the body, while outside it the flow tends to be steady. This phenomenon was studied experimentally by Taylor (1923), whose observations were confirmed theoretically by Grace (1926), Stewartson (1952, 1953, 1958), Bretherton (1967) and Miles (1975). Their theoretical work indicates that a body that starts moving slowly along the axis of rotation at a uniform velocity, ultimately has a stagnant column of fluid. The Taylor-column is set up ahead of the body and is bounded by the cylinder circumscribing the body and having its generators parallel to the axis of rotation.

In this paper, we have studied the flow created by the motion of a spheroid along the axis of an unbounded, inviscid, incompressible fluid rotating with uniform angular velocity. The pressure and velocity components at any point in space and on the surface of the spheroid have been obtained and ultimate velocity distribution for various cases have been discussed. The upstream flow due to oblate and prolate spheroids has been compared.

2. FUNDAMENTAL EQUATIONS

We consider the motion generated in an inviscid, incompressible unbounded rotating fluid when a spheroid impulsively starts to move along the axis of rotation at $t = 0$ with a uniform velocity $-V$. We take cylindrical polar coordinates with

x-axis along the axis of rotation and (r, θ) polar coordinates in a plane normal to OX. The fluid is in a state of rigid body rotation with constant angular velocity Ω about the axis of symmetry $r = 0$. If we choose the origin of coordinates at the centre of the spheriod we have in fact superposed uniform velocity $+V$ on the system and brought the spheroid to rest.

Let the components of fluid velocity along the directions of (x, r, θ) be $u, v, \Omega r + w$ respectively. The perturbations in the velocities due to the motion of the spheroid be sufficiently small so that their squares and products may be neglected.

The equations of motion, under no forces other than the pressure of the liquid, reduce to

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial P}{\partial x} \\ \frac{\partial v}{\partial t} - 2\Omega w &= -\frac{\partial P}{\partial r} \\ \frac{\partial w}{\partial t} + 2\Omega v &= 0 \end{aligned}\right\} \qquad \ldots(2.1)$$

where $$P = \frac{p}{\rho} - \tfrac{1}{2}\Omega^2 r^2. \qquad \ldots(2.2)$$

p, ρ represent pressure and density respectively of the fluid.

The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial x} = 0. \qquad \ldots(2.3)$$

The motion is symmetrical about the axis of rotation and hence is independent of θ.

We consider cases of oblate and prolate spheroids separately.

OBLATE SPHEROID

3. Integral Form of the Solution

Let the equation of the oblate spheroid be

$$\frac{x^2}{b^2} + \frac{r^2}{a^2} = 1,\ b < a \qquad \ldots(3.1)$$

The boundary conditions are

$$\left.\begin{aligned} &u \to V,\ v \to 0,\ w \to 0 \text{ as } x \to \infty \text{ for fixed } r \text{ and } t \\ &\frac{xu}{b^2} + \frac{rv}{a^2} = 0 \text{ on } \frac{x^2}{b^2} + \frac{r^2}{a^2} = 1 \text{ for all } t \\ &u = V,\ v = 0,\ w = 0 \text{ when } t = 0 \text{ for all } r,\ \frac{x^2}{b^2} + \frac{r^2}{a^2} > 1. \end{aligned}\right\} \qquad \ldots(3.2)$$

Applying Laplace transform defined by

$$\bar{u} = \int_0^\infty e^{-st}\, u(r, x, t)\, dt, \text{ etc.}$$

on eqns. (2.1), (2.3) and (3.2), the transformed velocity components in terms of pressure are

$$\left.\begin{aligned} \bar{n} &= \frac{V}{s} - \frac{1}{s}\frac{\partial \bar{P}}{\partial x} \\ \bar{v} &= -\frac{s}{s^2 + 4\Omega^2}\frac{\partial \bar{P}}{\partial r} \\ \bar{w} &= \frac{2\Omega}{s^2 + 4\Omega^2}\frac{\partial \bar{P}}{\partial r} \end{aligned}\right\} \qquad \ldots(3.3)$$

with boundary conditions

$$\left.\begin{aligned} &\frac{\partial \bar{P}}{\partial x} \to 0 \text{ as } x \to \infty \\ \text{and} \quad & \\ &xa^2 \frac{\partial \bar{P}}{\partial x} + \frac{s^2 r b^2}{s^2 + 4\Omega^2}\frac{\partial \bar{P}}{\partial r} = xVa^2 \quad \text{when } \frac{x^2}{b^2} + \frac{r^2}{a^2} = 1. \end{aligned}\right\} \qquad \ldots(3.4)$$

The equation of continuity reduces to

$$\frac{\partial^2 \bar{P}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{P}}{\partial r} + \frac{1 + \mu^2}{\mu^2}\frac{\partial^2 \bar{P}}{\partial x^2} = 0,$$

where $\mu = s/2\Omega$.

We now introduce oblate spheroidal coordinates ξ, η defined by

$$x = \frac{2\Omega b \eta \xi}{s}, \; r = \frac{2\Omega a}{(s^2 + 4\Omega^2)^{1/2}} (1 - \eta^2)^{1/2} (\xi^2 + 1)^{1/2}. \qquad \ldots(3.5)$$

Eliminating η from (3.5), we have

$$2\xi^2 = \left(\frac{r^2}{a^2} + \frac{x^2}{b^2}\right)\mu^2 + \left(\frac{r^2}{a^2} - 1\right) + \left[\left\{\left(\frac{r^2}{a^2} + \frac{x^2}{b^2}\right)\mu^2 + \left(\frac{r^2}{a^2} - 1\right)\right\}^2 + \frac{4x^2\mu^2}{b^2}\right]^{1/2} \qquad \ldots(3.6)$$

where $\mu = s/2\Omega$.

We make ξ one-valued by requiring it to be positive when s is large and positive.

When $x < 0, \mu < 0$; when $\frac{x^2}{b^2} + \frac{r^2}{a^2} = 1, \xi = \frac{s}{2\Omega}$; $\xi \to \infty$ when $x \to \infty$ for fixed r.

Therefore

$$\left.\begin{aligned} \frac{\partial \bar{P}}{\partial \xi} &= \frac{x}{\xi}\frac{\partial \bar{P}}{\partial x} + \frac{r\xi}{\xi^2+1}\frac{\partial \bar{P}}{\partial r} \\ \text{and}\quad \frac{\partial \bar{P}}{\partial \eta} &= \frac{x}{\eta}\frac{\partial \bar{P}}{\partial x} - \frac{r\eta}{(1-\eta^2)}\frac{\partial \bar{P}}{\partial r} \end{aligned}\right\} \qquad \ldots(3.7)$$

Hence in terms of ξ, η the continuity equation reduces to

$$\frac{\partial}{\partial \xi}\left[\{\xi^2+1\}\frac{\partial \bar{P}}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\{1-\eta^2\}\frac{\partial \bar{P}}{\partial \eta}\right] = 0. \qquad \ldots(3.8)$$

Comparing (3.4) and (3.7), we have

$$\frac{\partial \bar{P}}{\partial \xi} = \frac{xV}{\xi} = \frac{b\eta V}{\mu}. \qquad \ldots(3.9)$$

An appropriate solution of (3.8) is

$$\bar{P} = A\xi\eta + B\eta\left(\xi \log\frac{\xi - i}{\xi + i} + 2i\right) \qquad \ldots(3.10)$$

with boundary conditions

$$\frac{\partial \bar{P}}{\partial \xi} \to 0 \text{ as } \xi \to \infty \Rightarrow A = 0$$

and, on the surface of the spheroid $\xi = s/2\Omega = \mu$, we have

$$B\left[\log\frac{s-2\Omega i}{s+2\Omega i} + \frac{4\Omega s i}{s^2+4\Omega^2}\right] = bV\frac{2\Omega}{s}.$$

Applying inversion formula, we have

$$P = \frac{xV}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}\left[\log\frac{\xi-i}{\xi+i} + \frac{2i}{\xi}\right]ds}{\left[\log\frac{s-2\Omega i}{s+2\Omega i} + \frac{4s\Omega i}{s^2+4\Omega^2}\right]}. \qquad \ldots(3.11)$$

4. Force Along the Axis of Rotation

The resultant pressure thrust along the axis of rotation is given by

$$\chi = -\iint p\,\frac{x}{a}\,dS \text{ taken over the surface of the spheroid.}$$

Therefore

$$\chi = -\frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}\left[\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega}{s}\right]ds}{\left[\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4i\Omega s}{s^2+4\Omega^2}\right]}\iint \rho\frac{x^2}{a}\,dS \quad \ldots(4.1)$$

$$= \frac{2\rho a^2 bV}{3i}\int_{c-i\infty}^{c+i\infty} e^{st}\left[\frac{1}{2}+\frac{3}{10}\left(\frac{2i\Omega}{s}\right)^2+\frac{12}{175}\left(\frac{2i\Omega}{s}\right)^4+\ldots\right]ds \quad \ldots(4.2)$$

where c is sufficiently large positive quantity.

The first term on the right hand side is the impulsive pressure on the spheroid due to the initial rapid accelaration and is zero for $t > 0$.

On inversion, we have,

$$\chi = \frac{8\rho a^2 bV\Omega\pi}{3}\left[-\frac{3}{10}(2\Omega)\,t+\frac{12}{175}(2\Omega)^3\frac{t^3}{\underline{|3}}+\ldots\right]. \quad \ldots(4.3)$$

Note : When $b = a$, the spheroid becomes a sphere and the expression for χ is the same as that obtained by Stewartson (1952) and when Ωt is small it is in agreement with Grace (1926).

5. Velocity Components

The expressions for velocity components are obtained by using inversion formula on (3.3).

$$u = V - \frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{sN_2}\left[\log\frac{\xi-i}{\xi+i}+\frac{2i}{\xi}-\frac{2x^2s^2i}{b^2\xi^3N_1}\right]ds \quad \ldots(5.1)$$

where

$$N_1 = \left[\left\{\left(\frac{r^2}{a^2}+\frac{x^2}{b^2}\right)s^2+\left(\frac{s^2}{a^2}-1\right)4\Omega^2\right\}^2+\frac{16x^2s^2\Omega^2}{b^2}\right]^{1/2}$$

and

$$N_2 = \left[\log\frac{s-2i\Omega}{s+2i\Omega}+\frac{4s\Omega i}{s^2+4\Omega^2}\right].$$

$$v = \frac{V}{2\pi i a^2}\int_{c-i\infty}^{c+i\infty}\frac{2ixr\,e^{st}s\,ds}{\xi(\xi^2+1)\,N_1N_2} \quad \ldots(5.2)$$

$$w = -\frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{4ixr\,\Omega e^{st}ds}{a^2\xi(\xi^2+1)\,N_1N_2}. \quad \ldots(5.3)$$

6. Velocity Field on the Surface of the Spheroid

Here $\xi = \frac{s}{2\Omega}\cdot$ Therefore

$$u = V - \frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{sN_2}\left[\log\frac{s-2i\Omega}{s+2i\Omega} + \frac{4i\Omega}{s} - \frac{16\Omega^3x^2i}{s(s^2b^2+4\Omega^2x^2)}\right]ds. \quad ...(6.1)$$

The asymptotic form of u as $t \to \infty$ is quite interesting. The contribution from branch points at $s = \pm 2i\Omega$ tend to zero as $\Omega t \to \infty$. Only the poles at $s = 0$ and $s = \pm 2i\Omega x/b$ give contribution to u. Calculating the residues at the poles $s = 0$, $\pm 2i\Omega x/b$, we have

$$u \sim \frac{\frac{b}{\Omega x}\left[2\Omega b\left(\log\frac{b+x}{b-x} + \frac{2bx}{b^2-x^2}\right)\cos\frac{2\Omega bt}{x} - \left\{\pi^2 + \left(\log\frac{b+x}{b-x} + \frac{2bx}{b^2-x^2}\right)\left(\log\frac{b+x}{b-x} - \frac{2b}{x}\right)\sin\frac{2\Omega xt}{b}\right\}\right]}{\left[\pi^2 + \left(\log\frac{b+x}{b-x} + \frac{2bx}{b^2-x^2}\right)^2\right]}.$$

Since $(xu/b^2) + (rv/a^2) = 0$, on the spheroid, the variation of v on the surface of the spheroid can be obtained by substituting the value of u in $v = -xua^2/rb^2$.

Similarly $$w \sim -\frac{2rb^2V}{xa^2\pi}\left[1 + \frac{b^3\pi\left(\log\frac{b+x}{b-x} + \frac{2xb}{b^2-x^2}\right)\sin\frac{2\Omega xt}{b}}{(b^2-x^2)\left[\pi^2 + \left(\log\frac{b+x}{b-x} + \frac{2xb}{b^2-x^2}\right)^2\right]}\right].$$

7. Ultimate Velocity Distribution

If we replace the contour of integration of (4.1) by an infinite semicircle and contours round the branch points at $s = \pm 2i\Omega$, then it can be shown that in general the only contribution to u which does not tend to zero as $\Omega t \to \infty$ is that from the pole at $s = 0$. From (3.6), when $s = 0$

$$\xi = \frac{(r^2 - a^2)^{1/2}}{a}, \quad r > a$$

$$= 0, \qquad r \leqslant a.$$

We shall now consider four cases according as (i) $r > a$, when ξ is real and positive at $s = 0$; (ii) $r = a$ when $\xi = 0$ and the pole at $s = 0$ becomes a branch point; (iii) $r < a$, when $\xi = 0$ at $s = 0$ and (iv) $r = 0$, when the integral degenerates.

(i) $r > a$. $\xi = \frac{(r^2 - a^2)^{1/2}}{a}$ when $s = 0$.

Therefore

$$u = V + \frac{2V}{\pi}\left[\sin^{-1}\frac{a}{r} - \frac{a}{(r^2 - a^2)^{1/2}}\right].$$

Since $\xi \neq 0$ when $s = 0$, we see from (3.6) that ξ never vanishes and the only contribution to v and w are from branch points, $s = \pm 2i\Omega$ and these tend to zero asymptotically. Hence ultimately v and w tend to zero when $r > a$ and $\Omega t \to \infty$.

(ii) $r = a$. When $s = 0$, $\xi = 0$; when s is small

$$\xi = \left(\frac{xs}{2\Omega b}\right)^{1/2} + O\left(\frac{s}{\Omega}\right).$$

Hence the pole at $s = 0$ in the integrand for u now becomes a branch point. Thus

$$u \sim -\frac{4V}{\pi}\left(\frac{2\Omega bt}{x\pi}\right)^{1/2}.$$

Hence the contribution of the branch point at $s = 0$, unlike those from other branch points, increases indefinitely.

The only possible non-zero contribution to v is from the branch point at $s = 0$. Near $s = 0$,

$$v \sim \frac{Vb}{\pi a}\left(\frac{b}{2\Omega xt\pi}\right)^{1/2}, \text{ and which tend to zero as } \Omega t \to \infty.$$

When s is small, the relevant contribution to the transverse velocity w is

$$-\frac{2Vb}{\pi a}\left(\frac{2\Omega bt}{\pi x}\right)^{1/2}.$$

Thus the transverse velocity is not ultimately zero on the cylinder when $\Omega t \to \infty$. The flow is therefore not small on the cylinder $r = a$, but since the radial velocity v is ultimately zero on the cylinder, there is no exchange of fluid between the regions inside and outside the cylinder.

Exceptions to this two-dimensional motion arise on the spheroid where the velocities oscillate finitely, and on the cyclinder $r = a$ where w and u increase indefinitely with $\Omega\, t$. It may be noted that the ultimate motion is independent of how it started.

(iii) $r < a$. In this case $\xi = 0$ when $s = 0$. When s is small,

$$\xi = \frac{sxa}{2\Omega b(a^2 + r^2)^{1/2}}.$$

Hence the integrand of (5.1) has a simple pole at $s = 0$ with residue 1 so that u is ultimately zero inside the cylinder except possibly at $r = 0$. Moreover, since the integrand of (5.2) is regular near $s = 0$, v also vanishes in the limit.

$$w = -\frac{2Vrb}{\pi a(a^2 - r^2)^{1/2}} \text{ as } \Omega t \rightarrow \infty.$$

Thus inside the cylinder the axial velocity of the fluid is ultimately the same as that of the spheroid but in addition there is a swirling motion about the axis with a singularity at $r = a$.

(iv) $r = 0$. On the axis of rotation $r = 0$, v and w are identically zero and hence there is only an axial velocity u.

$$\xi = xs/2\Omega b$$

$$\therefore\ u \sim \frac{2xV\left(\log\dfrac{x+b}{x-b} + \dfrac{2bx}{x^2-b^2}\right)\cos\dfrac{2\Omega bt}{x}}{\left[\pi^2 + \left(\log\dfrac{x+b}{x-b} + \dfrac{2bx}{x^2-b^2}\right)^2\right]}.$$

Thus we see that, in general, the ultimate motion is steady, two-dimensional and small, and a cylinder of radius s is pushed along in front of it. The fluid inside has an increased angular velocity, but the cylinder acts like an impossible barrier to make the flow inside and outside apparently independent.

PROLATE SPHEROID

8. Integral Form of the Solution

Let the equation of prolate spheroid be

$$\frac{x^2}{a^2} + \frac{r^2}{b^2} = 1, \quad a > b. \qquad \text{...(8.1)}$$

The boundary conditions are

$u \rightarrow V$, $v \rightarrow 0$, $w \rightarrow 0$ as $x \rightarrow \infty$ for fixed r and t.

$$\frac{xu}{a^2} + \frac{rv}{b^2} = 0 \text{ on } \frac{x^2}{a^2} + \frac{r^2}{b^2} = 1 \text{ for all } t. \qquad \text{...(8.2)}$$

$u = V$, $v = 0$, $w = 0$ when $t = 0$ for all r, x satisfying $\dfrac{x^2}{a^2} + \dfrac{r^2}{b^2} > 1$. The transformed boundary condition on the surface is

$$\frac{x\bar{u}}{a^2} + \frac{r\bar{v}}{b^2} = 0$$

which reduces to

$$b^2 x \frac{\partial \bar{P}}{\partial x} + \frac{s^2 r a^2}{s^2 + 4\Omega^2}\frac{\partial \bar{P}}{\partial r} = b^2 x V \qquad \text{...(8.3)}$$

on using the values of $\bar{u}$ and $\bar{v}$ from (3.3).

We now introduce prolate spheroidal coordinates ξ and η defined by

$$x = \frac{2\Omega a\eta\xi}{s}, \quad r = \frac{2\Omega b}{\sqrt{s^2 - 4\Omega^2}} (1 - \eta^2)^{1/2} (\xi^2 - 1)^{1/2}. \qquad \text{...(8.4)}$$

Eliminating η from (8.4), we have

$$2\xi^2 = \left(\frac{r^2}{b^2} + \frac{x^2}{a^2}\right)\mu^2 + \left(1 - \frac{r^2}{b^2}\right) + \left[\left\{\left(\frac{r^2}{b^2} + \frac{x^2}{a^2}\right)\mu^2 + \left(1 - \frac{r^2}{b^2}\right)\right\}^2 - \frac{4x^2\mu^2}{a^2}\right]^{1/2} \qquad \text{...(8.5)}$$

When $\frac{x^2}{a^2} + \frac{r^2}{b^2} = 1, \quad \xi = \frac{s}{2\Omega}$;

when $x < 0, \eta < 0$; $\xi \to \infty$ with $x \to \infty$ for fixed r.

In terms of ξ and η, the equation of continuity reduces to

$$\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial \bar{P}}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial \bar{P}}{\partial\eta}\right] = 0. \qquad \text{...(8.6)}$$

An appropriate solution of eqn. (8.6) is

$$\bar{P} = A\eta\xi + B\eta\left(\xi \log\frac{\xi - 1}{\xi + 1} + 2\right). \qquad \text{...(8.7)}$$

Applying the boundary conditions, we have $A = 0$,

$$B\left[\log\frac{s - 2\Omega}{s + 2\Omega} + \frac{4s\Omega}{s^2 - 4\Omega^2}\right] = aV\frac{2\Omega}{s}$$

Applying inversion formula, we have

$$P = \frac{xV}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}\left(\log\frac{\xi - 1}{\xi + 1} + \frac{2}{\xi}\right) ds}{\left(\log\frac{s - 2\Omega}{s + 2\Omega} + \frac{4s\Omega}{s^2 - 4\Omega^2}\right)}. \qquad \text{...(8.8)}$$

9. Force Along the Axis of Rotation

The resultant pressure is along the axis of rotation and is given by

$$\chi = -\iint p\frac{x}{b}\, ds \text{ taken over the surface of the spheroid.}$$

$$\chi = -\frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}\left[\log\frac{s - 2\Omega}{s + 2\Omega} + \frac{4\Omega}{s}\right] ds}{\left[\log\frac{s - 2\Omega}{s + 2\Omega} + \frac{4s\Omega}{s^2 - 4\Omega^2}\right]} \iint \rho\frac{x^2}{b}\, dS \qquad \text{...(9.1)}$$

$$= \frac{8\Omega\rho ab^2 V\pi}{3}\left[\frac{3}{10}(2\Omega)\,t + \frac{12}{175}(2\Omega)^3\frac{t^3}{\lfloor 3} + \frac{4}{125}(2\Omega)^5\frac{t^5}{\lfloor 5} + \ldots\right]. \qquad \ldots(9.2)$$

10. Velocity Components

The velocity components are obtained by using inversion formula :

$$u = V - \frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s}\left[\log\frac{\xi-1}{\xi+1} + \frac{2}{\xi} - \frac{2x^2s^2}{a^2\xi^3 N_3}\right]\frac{ds}{N_4} \qquad \ldots(10.1)$$

where

$$N_3 = \left[\left\{\left(\frac{r^2}{b^2} + \frac{x^2}{a^2}\right)s^2 + 4\Omega^2\left(1 - \frac{r^2}{b^2}\right)\right\}^2 - \frac{16\Omega^2x^2s^2}{a^2}\right]^{1/2}$$

and

$$N_4 = \left[\log\frac{s-2\Omega}{s+2\Omega} + \frac{4s\Omega}{s^2-4\Omega^2}\right].$$

$$v = -\frac{V}{2\pi i b^2}\int_{c-i\infty}^{c+i\infty}\frac{2xr\,e^{st}\,s(s^2-4\Omega^2)\,ds}{(s^2+4\Omega^2)\,\xi(\xi^2-1)\,N_3\,N_4}. \qquad \ldots(10.2)$$

$$w = \frac{V}{2\pi i b^2}\int_{c-i\infty}^{c+i\infty}\frac{4\Omega xr\,e^{st}\,(s^2-4\Omega^2)\,ds}{(s^2+4\Omega^2)\,\xi(\xi^2-1)\,N_3\,N_4}. \qquad \ldots(10.3)$$

11. Velocity Field on the Surface of the Spheroid

Here $\xi = \dfrac{s}{2\Omega}$.

$$\therefore\; u = V - \frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{st}}{s^2}\left[s\log\frac{s-2\Omega}{s+2\Omega} + 4\Omega - \frac{16x^2\Omega^3}{(s^2a^2-4\Omega^2x^2)}\right]$$

$$\times\frac{ds}{\left[\log\dfrac{s-2\Omega}{s+2\Omega} + \dfrac{4s\Omega}{s^2-4\Omega^2}\right]}$$

Taking contribution from the poles at $s = 0, \pm 2\Omega x/a$, we have

$$u = \frac{16}{\pi^2} - \frac{a}{\Omega x}\left[\frac{\left[\pi^2 + \left\{\log\dfrac{a+x}{a-x}\right\}^2 + \dfrac{2ax}{a^2-x^2}\log\dfrac{a+x}{a-x} + \dfrac{2a(2ax)}{x(a^2-x^2)} - \dfrac{2a}{x}\log\dfrac{a+x}{a-x}\right]\times\sin h\left(\dfrac{2\Omega xt}{a}\right) + 2\Omega a\left(\log\dfrac{a+x}{a-x} + \dfrac{2ax}{a^2-x^2}\right)\cos h\left(\dfrac{2\Omega xt}{a}\right)}{\left[\pi^2 + \left(\log\dfrac{a+x}{a-x} + \dfrac{2ax}{a^2-x^2}\right)^2\right]}\right].$$

The value of v on the surface of the spheroid can be obtained by substituting the value of u in $v = -xub^2/a^2r$.

$$w \sim -\frac{2ra^2V}{xb^2(x^2+a^2)} \times \left[\frac{2x^2 \sin 2\Omega t}{(3\pi+2)} + \frac{a^3\left\{\log\frac{a+x}{a-x} + \frac{2ax}{a^2-x^2}\right\} \sin h\,\frac{2\Omega xt}{a}}{\left\{\pi^2 + \left(\log\frac{a+x}{a-x} + \frac{2ax}{a^2-x^2}\right)^2\right\}}\right].$$

This shows that the motion is not ultimately steady on the spheroid.

12. Ultimate Velocity Distribution

(i) $r > b$. When $s = 0$, $\xi = 0$. Therefore taking s small, we have

$$\xi = \frac{ixsb^2}{2a\Omega(r^2-b^2)^{1/2}} + O\left(\frac{s^2}{b^2}\right).$$

$$\therefore\ u \sim \frac{V(r^2-b^2)^{1/2}\,4a\Omega t}{\pi xb}.$$

Since ξ vanishes when $s = 0$ the only contribution to v and w are from branch points and these tend to zero asymptotically. Hence ultimately v and w tend to zero when $r > b$ and $\Omega t \to \infty$.

(ii) $r = b$. When s is small $\xi = \left(\frac{ixs}{2\Omega a}\right)^{1/2}$

$$u = \frac{4V}{\pi}\left(\frac{\Omega at}{\pi x}\right)^{1/2}$$

$$v \sim \frac{aV}{2b\pi}\left(\frac{a}{\Omega xt\pi}\right)^{1/2}$$

$$w \sim -\frac{2aV}{b\pi}\left(\frac{a\Omega t}{\pi x}\right)^{1/2}.$$

(iii) $r < b$. When $s = 0$, $\xi = \frac{(b^2-r^2)^{1/2}}{b}$

$$u = V - \frac{V}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s}\left[\log\frac{(b^2-r^2)^{1/2}-b}{(b^2-r^2)^{1/2}+b} + \frac{2b}{(b^2-r^2)^{1/2}}\right]\frac{ds}{N_4}$$

$$u = V.$$

Since the integrand of eqn. (10.2) is regular at $s = 0$, v vanishes in the limit. Hence the only contribution is obtained from the angular velocity w.

$$w = -\frac{ixb^3\,\delta(t)}{r(b^2 - r^2)^{1/2}\Omega\pi}$$

where $\delta(t)$ is a Dirac Delta function and hence zero for all $t > 0$. Thus in this case the fluid inside the cylinder is stagnant and is pushed along the axis of rotation with velocity V of the spheroid.

(iv) $r = 0$. Since on the axis of rotation $r = 0$, v, w both identically vanish and hence there is only an axial velocity u.

Here $\xi = \frac{xs}{2\Omega a}$.

$$\therefore\ u \sim \frac{16aV}{\pi^2 x} - \frac{2xV\left(\log\frac{x+a}{x-a} + \frac{2ax}{x^2-a^2}\right)}{\left[\pi^2 + \left(\log\frac{x+a}{x-a} + \frac{2ax}{x^2-a^2}\right)^2\right]}.$$

In general the ultimate flow is steady, two dimentional and small, and a liquid cylinder of radius b is pushed along in front of it.

13. Conclusion

Thus, inside the cylinder, the ultimate flow consists of a uniform translation along the direction of the motion of the spheroid. In both cases the liquid cylinder is pushed along the axis of rotation, with a greater force in the case of prolate spheroid. In the case of oblate spheroid there is a swirling motion about the axis of cylinder while for prolate spheroid the liquid is stagnant inside the cylinder. Also it may be observed that the ultimate motion outside the cylinder is steady in case of oblate spheroid and time dependent for the prolate. Outside the cylinder there is only a shearing motion parallel to the axis of rotation, singular on the surface of the cylinder and tending to zero as $r \to \infty$. For the oblate spheroid the projection of the streamlines on y - z plane are circles, and since none of them meets the cylinder no fluid is transferred from the region inside the cylinder to the region outside the cylinder. This is also obvious from the fact that due to the clusterings of the branch points near $s = 0$ on the surface of the cylinder the flow takes much longer time to achieve.

As a limiting case the ultimate flow due to a very elongated spheroid (perpendicular to the axis) is similar to that of disc.

It may be noted that the results for the oblate spheroid are in accordance with those obtained by Stewartson (1953, Art 5).

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