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ON NEO-PSEUDO PROJECTIVE TENSOR FIELDS

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In the present paper the authors define a tensor field by taking linear combination of the projective deviation tensor field W^i_j and pseudo derivation tensor field T^i_j (Sinha 1971). Some properties and certain theorems of this tensor field are discussed.

1. INTRODUCTION

Let us consider an n-dimensional Finsler space F_n equipped with Berwald's connection coefficients $G^i_{jk}(x, \dot{x})$ (Rund 1959).

Let T^i_j be any tensor field. The following commutation formulae will be used in sequel (Rund 1959)

$$\left.\begin{aligned} &\text{(a)} \quad \left(\frac{\partial T^i_j}{\partial \dot{x}^k}\right)_{(h)} - \frac{\partial T^i_{j(h)}}{\partial \dot{x}^k} = T^i_r\, G^r_{jkh} - T^r_j\, G^i_{rkh} \\ &\text{(b)} \quad \left(\frac{\partial T^i_{jl}}{\partial \dot{x}^k}\right)_{(h)} - \frac{\partial T^i_{jl(h)}}{\partial \dot{x}^k} = T^i_{rl}\, G^r_{jkh} + T^i_{jr}\, G^r_{lkh} - T^r_{jl}\, G^i_{rkh} \end{aligned}\right\} \quad \ldots(1.1)$$

where $\quad G^i_{jkh} = \dot{\partial}_j G^i_{kh}$.

The projective deviation tensor field $W^i_k(x, \dot{x})$ is given by

$$W^i_k = H^i_k - H\delta^i_k - (\dot{\partial}_i H^i_k - \dot{\partial}_k H)\frac{\dot{x}^i}{n+1} \quad \ldots(1.2)$$

where $H^i_k(x, \dot{x})$ is geodesic deviation tensor field (Rund 1959).

The tensor field $W^i_k(x, \dot{x})$ yields the projective curvature tensor fields :

$$\text{(a)} \quad W^i_{hk} = \tfrac{2}{3}\,\dot{\partial}_{[h} W^i_{k]}, \quad \text{(b)} \quad W^i_{ihk} = \dot{\partial}_i W^i_{hk} \quad \ldots(1.3)$$

$\dot{\partial}_i = \dfrac{\partial}{\partial \dot{x}^i}$.

The projective deviation tensor and the projective curvature tensor fields satisfy the identities (Rund 1959) :

$$\text{(a)} \quad W^j_k \, \dot{x}^k = 0, \quad \text{(b)} \quad W^j_j = 0, \quad \text{(c)} \quad W^j_{hj} = 0, \quad \text{(d)} \quad W^j_{ihj} = 0. \qquad \ldots(1.4)$$

We note here that $W^j_k(x, \dot{x})$ is homogeneous of degree two in its directional arguments.

The pseudo deviation tensor field defined by Sinha (1971) is given by

$$T^j_k = -\left[H\delta^j_k + \frac{1}{(n+1)}(\dot{\partial}_i H^i_k - \dot{\partial}_k H)\,\dot{x}^j\right]. \qquad \ldots(1.5)$$

In analogy with tensors W^j_{hk}, W^j_{ihk} the pseudo-curvature tensor fields $T^j_{hk}(x, \dot{x})$ and $T^j_{ihk}(x, \dot{x})$ are defined by

$$\text{(a)} \quad T^j_{hk} = \tfrac{2}{3}\,\dot{\partial}_{[h} T^j_{k]} \quad \text{and} \quad \text{(b)} \quad T^j_{ihk} = \dot{\partial}_i T^j_{hk}. \qquad \ldots(1.6)$$

The pseudo deviation tensor field $T^j_k(x, \dot{x})$ is also homogeneous of degree two in its directional arguments and satisfies the relation (Sinha 1971)

$$T^j_k \, \dot{x}^k = 0. \qquad \ldots(1.7)$$

On contracting the pseudo deviation tensor and pseudo curvature tensor fields, we have the following relations :

$$\text{(a)} \quad T^j_j = (n-1)\,T, \quad \text{(b)} \quad T^j_{hj} = T_h, \quad \text{(c)} \quad T^j_{ihj} = T_{ih}. \qquad \ldots(1.8)$$

Also, we have

$$\dot{\partial}_i T_j = T_{ij}. \qquad \ldots(1.9)$$

2. Neo-Pseudo Projective Tensor Fields

Considering the linear combination of the projective deviation tensor field W^i_j and the pseudo deviation tensor field T^i_j, we have

$$Q^i_j \overset{def}{=} pW^i_j + qT^i_j \qquad \ldots(2.1)$$

where p and q are scalar functions of $(x, \dot{x})$ and homogeneous of degree zero in $\dot{x}^i$. We call Q^i_j the neo-pseudo projective tensor field.

Contracting the indices i and j in (2.1) and using (1.4b) and (1.8a) we have

$$Q^i_i = q(n-1)\,T. \qquad \text{...(2.2)}$$

Multiplying (2.1) by $\dot{x}^j$ and using eqns. (1.4a) and (1.7a), we get

$$Q^i_j\,\dot{x}^j = 0 \qquad \text{...(2.3)}$$

which after partial differentiation with respect to $\dot{x}^h$, gives

$$\dot{\partial}_h Q^i_j\,\dot{x}^j = -\,Q^i_h\,. \qquad \text{...(2.4)}$$

We define the neo-pseudo projective curvature tensor fields as

$$Q^i_{hj}\,(x, \dot{x}) = \tfrac{2}{3}\,\dot{\partial}_{[h}\,Q^i_{j]} \qquad \text{...(2.5)}$$

$$Q^i_{lhj}\,(x, \dot{x}) = \dot{\partial}_l Q^i_{hj} = \tfrac{2}{3}\,\dot{\partial}^2_{l[h}\,Q^i_{j]}\,. \qquad \text{...(2.6)}$$

The tensor fields Q^i_{hj} and Q^i_{lhj} can be expressed in the forms

$$Q^i_{hj} = pW^i_{hj} + qT^i_{hj} + \tfrac{2}{3}\,[\dot{\partial}_{[h}\,pW^i_{j]} + \dot{\partial}_{[h}qT^i_{j]}]. \qquad \text{...(2.7)}$$

and

$$Q^i_{lhj} = pW^i_{lhj} + qT^i_{lhj} + (\dot{\partial}_l p)\,W^i_{hj} + (\dot{\partial}_l q)\,T^i_{hj} + \tfrac{2}{3}\,[\dot{\partial}^2_{l[h}\,pW^i_{j]}$$

$$+\,\dot{\partial}_{[h}\,p\dot{\partial}_{|l|}\,W^i_{j]} + \dot{\partial}^2_{l[h}\,qT^i_{j]} + \dot{\partial}_{[h}\,q\dot{\partial}_{|l|}\,T^i_{j]}]. \qquad \text{...(2.8)}$$

By virtue of homogeneity properties of T^i_j, W^i_j, p and q we note that the neo-pseudo projective deviation tensor Q^i_j is also positively homogeneous of degree two in its directional arguments.

Transvecting (2.5) by $\dot{x}^h$ and (2.6) by $\dot{x}^l$ and using (2.4), we have

$$Q^i_{hj}\,\dot{x}^h = Q^i_j\,,\quad Q^i_{lhj}\,\dot{x}^l = Q^i_{hj}$$

which gives

$$Q^i_{lhj}\,\dot{x}^l\dot{x}^h = Q^i_j\,.$$

Contracting (2.7) and (2.8) with respect to indices i, j and simplifying with the help of (1.4), (1.8), (1.9), we have

The index inside " | | " is free from skew-symmetric part.

$$Q^i_{hi} = qT_h + \tfrac{1}{3}\,[(\dot{\partial}_h q)\,(n-1)\,T - (\dot{\partial}_i p)\,W^i_h - (\dot{\partial}_i q)\,T^i_h] \qquad ...(2.9)$$

and

$$Q^i_{lhi} = qT_{lh} + (\dot{\partial}_l q)\,T_h + \tfrac{1}{3}\,[(n-1)\,(T\dot{\partial}^2_{hl}\,q - \dot{\partial}_h q\dot{\partial}_l T)$$

$$- \{(\dot{\partial}^2_{li}\,p)\,W^i_h + \dot{\partial}_i p\dot{\partial}_l W^i_h\} - \{(\dot{\partial}^2_{li}\,q)\,T^i_h + \dot{\partial}_i q\dot{\partial}_l T^i_h\}]. \qquad ...(2.10)$$

Now we shall prove the following theorems :

Theorem 2.1 — The Bianchi identities for the neo-pseudo projective curvature tensor fields Q^i_{hj} and Q^i_{lhj} are

$$^*Q^i_{\{hj(s)\}} = \tfrac{1}{3}\,(\dot{\partial}_{\{h}\,Q^i_{j(s)\}} - \dot{\partial}_{\{j}\,Q^i_{h(s)\}}). \qquad ...(2.11)$$

and

$$Q^i_{l\{hj(s)\}} = \dot{\partial}_l Q^i_{\{hj(s)\}} - Q^r_{\{hj}\,G^i_{s\}rl} \text{ respectively.} \qquad ...(2.12)$$

PROOF : Differentiating (2.5) covariantly with respect to $\dot{x}^s$ and making use of commutation formula (1.1*a*), we get

$$Q^i_{hj(s)} = \tfrac{1}{3}\,[\dot{\partial}_h Q^i_{j(s)} - \dot{\partial}_j Q^i_{h(s)} - Q^r_j\,G^i_{hrs} + Q^r_h\,G^i_{jrs}]. \qquad ...(2.13)$$

Adding the expressions obtained by cyclic permutation of the indices h, j and s in above expression, we get (2.11).

Differentiating (2.6) covariantly with respect to x^s and making use of the commutation formula (1.1*b*), we get

$$Q^i_{lhj(s)} = \dot{\partial}_l Q^i_{hj(s)} + Q^i_{rj}\,G^r_{hls} + Q^i_{hr}\,G^r_{jls} - Q^r_{hj}\,G^i_{rls}. \qquad ...(2.14)$$

Adding the expressions obtained by cyclic permutation of the indices h, j and s in (2.14), we get (2.12).

Theorem 2.2 — The neo-pseudo projective curvature tensor field Q^i_{lhj} satisfies the following identity

$$Q^i_{lhj(s)} + Q^i_{jls(h)} + Q^i_{sjh(l)} + Q^i_{hsl(j)} = \dot{\partial}_l Q^i_{hj(s)}$$

$$+ \dot{\partial}_j Q^i_{ls(h)} + \dot{\partial}_s Q^i_{jh(l)} + \dot{\partial}_h Q^i_{sl(j)}. \qquad ...(2.15)$$

$$^*Q^i_{\{hj(s)\}} \overset{def}{=} Q^i_{hj(s)} + Q^i_{js(h)} + Q^i_{sh(j)}$$

PROOF : Interchanging l, j, s, h cyclically in (2.14) and adding the expressions thus obtained with (2.14) and noting that Q^i_{hj} is skew-symmetric in h and j, we get (2.15).

Theorem 2.3 — The neo-pseudo projective curvature tensor field Q^i_{lhj} satisfies the following identities:

$$\text{(a)} \quad Q^{i*}_{\{lhj\}} = 0 \quad \text{and} \quad \text{(b)} \quad Q^i_{[l\,|\,h\,|\,j]} = \tfrac{1}{3}\,\dot{\partial}^2_{h[l}\, Q^i_{j]}\,. \qquad ...(2.16)$$

PROOF : The proof follows from Theorem 2.1.

Theorem 2.4 — The tensor field $Q_{lkhj} \overset{def}{=} g_{ik} Q^i_{lhj}$ satisfies the identities :

$$\text{(a)} \quad Q_{lkhj} + Q_{kljh} = \tfrac{2}{3}\,[g_{ik}\dot{\partial}^2_{l[h}\, Q^i_{j]} + g_{il}\dot{\partial}^2_{k[j}\, Q^i_{h]}]$$

$$\text{(b)} \quad Q_{lkhj} + Q_{jhkl} = \tfrac{2}{3}\,[g_{ik}\dot{\partial}^2_{l[h}\, Q^i_{j]} + g_{ih}\dot{\partial}^2_{j[k}\, Q^i_{l]}]$$

$$\text{(c)} \quad Q_{lkhj} - Q_{ljhk} + Q_{hjlk} - Q_{hklj} = \tfrac{2}{3}\,[g_{i[k}\dot{\partial}^2_{j]l}\, Q^i_h + g_{i[k}\dot{\partial}^2_{j]h}\, Q^i_l\,]$$

$$\text{(d)} \quad Q_{lkhj} + Q_{kljh} + Q_{jhkl} + Q_{hjlk} = \tfrac{2}{3}\,[g_{ik}\dot{\partial}^2_{l[h}\, Q^i_{j]} + g_{il}\dot{\partial}^2_{k[j}\, Q^i_{h]}$$
$$+ g_{ih}\dot{\partial}^2_{j[k}\, Q^i_{l]} + g_{ij}\dot{\partial}^2_{h[l}\, Q^i_{k]}]. \qquad ...(2.17)$$

PROOF : Multiplying (2.6) by g_{ik}, we have

$$Q_{lkhj} = \tfrac{2}{3}\, g_{ik}\dot{\partial}^2_{l[h}\, Q^i_{j]}\,. \qquad ...(2.18)$$

from which we obtain the results.

3. RECURRENT FINSLER SPACES

A recurrent Finsler space F_n is characterised by the relation

$$H^i_{jkh(l)} = v_l H^i_{jkh} \qquad ...(3.1)$$

where v_l is non-zero recurrence vector field and H^i_{jkh} is Berwald's curvature tensor field.

It has been shown by Sinha and Singh (1971) that in a recurrent Finsler space H^i_j , H^i_{jk}, W^i_j and W^i_{jk} are recurrent, each having the recurrence vector field v_l.

$^*Q^i_{\{lhj\}} \overset{def}{=} Q^i_{lhj} + Q^i_{hjl} + Q^i_{jlh}$

The relation $T^i_j = W^i_j - H^i_j$ shows that the tensor T^i_j will also be recurrent having same recurrence vector field v_l. It may, however, be noted that in a recurrent Finsler space the neo-pseudo projective deviation tensor field Q^i_j is not, in general, recurrent.

In fact, the relation (2.1) and facts that W^i_j and T^i_j are recurrent. yield

$$Q^i_{j(l)} = v_l Q^i_j + p_{(l)} W^i_j + q_{(l)} T^i_j \qquad \text{...(3.2)}$$

The following theorems are immediate consequences of eqn. (3.2).

Theorem 3.1 — The sufficient condition that the neo-pseudo projective deviation tensor field Q^i_j be recurrent is that the scalars p and q are covariant constants.

Theorem 3.2 — In a recurrent Finsler space if the neo-pseudo projective deviation tensor field Q^i_j is recurrent then the projective deviation and pseudo deviation tensor fields are proportional at each point of the space.

Remark

Case 1 — $p = 1$, $q = 0$, the neo-pseudo projective deviation tensor field Q^i_j coincides with the projective deviation tensor field W^i_j and its properties have been discussed. (see Rund 1959).

Case 2 — $p = 0$, $q = 1$, the neo-pseudo projective deviation tensor field Q^i_j coincides with the pseudo deviation tensor field T^i_j and its properties have been given by Sinha (1971).

Case 3 — $p = 1$, $q = 1$, then we have

$$Q^i_j = W^i_j - T^i_j = H^i_j \cdot$$

Thus neo-pseudo projective deviation tensor field Q^i_j coincides with the deviation tensor field and its properties have been given in Rund (1959).

References

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