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MULTIPLE FINITE SPHEROIDAL TRANSFORM AND ITS APPLICATION*

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In the present paper the multiple finite spheroidal transform has been developed and its operational property has been studied. As an application of this transform, we have considered the laminar forced convection of a fluid inside an infinitely long channel of square cross-section, the source of heat generation and the conductivity of the fluid being variable.

1. INTRODUCTION

Recently, Gupta (1974) has developed the generalized Gegenbauer transform by the equation

$$G\{f(x)\} \equiv \bar{f}_\alpha(c, n) = \int_{-1}^{1} (1 - x^2)^\alpha \psi_{\alpha n}(c, x) f(x) \, dx \qquad \ldots(1)$$

$$(n = 0, 1, \ldots; \alpha > -1)$$

where the spheroidal function, $\psi_{\alpha n}(c, x)$, satisfies the differential equation [Rhodes 1970, eqn. (1)] :

$$(1 - x^2) \psi''_{\alpha n}(c, x) - 2(\alpha + 1) x \psi'_{\alpha n}(c, x) + [b_{\alpha n}(c) + c^2 x^2] \psi_{\alpha n}(c, x) = 0 \qquad \ldots(2)$$

$b_{\alpha n}(c)$ being the separation constants for every finite value of c and $n = 0, 1, \ldots$. The function $\psi_{\alpha n}(c, x)$ can be expanded as

$$\sum_{k=0,1}^{\infty}{}^{\dagger} \; d_k(c \mid \alpha n) \, T_k^\alpha(x). \qquad \ldots(3)$$

where the asterisk '†' over the summation sign indicates that the summation is taken over only even or odd values of k according as n is even or odd. The coefficients $d_k(c \mid \alpha n)$ satisfy the three term recursion formula (Stratton *et al.* 1956, p. 13, eqn. (82)].

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The orthogonality relation on $(-1, 1)$ is designated by

$$\int_{-1}^{1} \psi_{\alpha n}(c, x)\, \psi_{\alpha p}(c, x)\, (1 - x^2)^\alpha \, dx = \Lambda_{\alpha n}(c)\, \delta_{np} \qquad \ldots(4)$$

where Λ_{np} denotes the normalization factor for the spheroidal function of order α and index n, given by [Rhodes 1970, eqn. (43)] :

$$\Lambda_{\alpha n}(c) = \sum_{k=0,1}^{\infty}{}^{*} \frac{k!}{(k + \alpha + \frac{1}{2})\, \Gamma(k + 2\alpha + 1)}\, a_k^2 (c \mid \alpha n) \qquad \ldots(5)$$

where the a_k's are the expansion coefficients given by [Rhodes 1970, eqn. (15)] :

$$a_k(c \mid \alpha n) = i^{k-n} \frac{\Gamma(k + 2\alpha + 1)}{k!}\, d_k(c \mid \alpha n). \qquad \ldots(6)$$

We shall require the following results [Morse 1953, p. 782, 783; Rhodes 1970, eqn. (8)] :

$$T_0^\alpha (x) = \frac{2^\alpha \Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}} \qquad \ldots(7)$$

$$T_1^\alpha (x) = \frac{2^{\alpha+1} x}{\sqrt{\pi}}\, \Gamma(\alpha + \tfrac{3}{2}) \qquad \ldots(8)$$

$$\frac{d}{dx} [(x^2 - 1)^\alpha\, T_n^\alpha (x)] = (n + 1)(n + 2\alpha)\, (x^2 - 1)^{\alpha-1}\, T_{n+1}^{\alpha-1} (x) \qquad \ldots(9)$$

$$\int_{-1}^{1} (1 - x^2)^\alpha\, T_n^\alpha (x)\, T_m^\alpha (x)\, dx = \delta_{mn} \frac{2\Gamma(n + 2\alpha + 1)}{(2n + 2\alpha + 1)\, \Gamma(n + 1)} \qquad \ldots(10)$$

$$\int_{-1}^{1} e^{ic\eta t}(1 - t^2)^\alpha\, T_n^\alpha (t)\, dt = \frac{\sqrt{(2\pi)}\, i^n\, \Gamma(n + 2\alpha + 1)}{n!} \times \frac{J_{n+\alpha+(1/2)}(c\eta)}{(c\eta)^{\alpha+(1/2)}},\ \alpha > -1. \qquad \ldots(11)$$

2. Representation Theorem and Inversion Formula

Theorem 1 — If $f(x, y)$ is continuous (and of bounded variation) over the square $\{(x, y) : -1 < x < 1, -1 < y < 1\}$, and if the finite spheroidal transform of $f(x, y)$ is defined by the equation

$$T\,[f(x, y); (x, y) \to (p, q)] \equiv \bar{f}(p, q) = \int_{-1}^{1} \int_{-1}^{1} (1 - x^2)^\alpha\, (1 - y^2)^\beta\, \psi_{\alpha n_1}(p, x)\, \psi_{\beta n_2}(q, y)\, f(x, y)\, dx\, dy \qquad \ldots(12)$$

then

$$T^{-1}[\bar{f}(p,q);(p,q)\to(x,y)]\equiv f(x,y)$$

$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\bar{f}(p,q)}{\Lambda_{\alpha n_1}(p)\,\Lambda_{\beta n_2}(q)}\,\psi_{\alpha n_1}(p,x)\,\psi_{\beta n_2}(q,y) \quad \ldots(13)$$

at the points of the square at which $f(x, y)$ is continuous.

PROOF: By generalized Fourier series (Churchill 1963), the function $f(x, y)$ can be expanded as

$$f(x,y)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}D_{n_1n_2}\psi_{\alpha n_1}(p,x)\,\psi_{\beta n_2}(q,y),\ (-1\leqslant x\leqslant 1,-1\leqslant y\leqslant 1) \quad \ldots(14)$$

By grouping the terms in double infinite series (14) so as to display the total coefficients of $\psi_{\alpha n_1}(p, x)$ for each n_1, we can write formally

$$f(x,y)=\sum_{n_1=0}^{\infty}\Big[\sum_{n_2=0}^{\infty}D_{n_1n_2}\psi_{\beta n_2}(q,y)\Big]\,\psi_{\alpha n_1}(p,x). \quad \ldots(15)$$

Normalizing the function $\psi_{\alpha n_1}(p, x)$ for all values of $p(n_1$ is even or odd), we get

$$\sum_{n_2=0}^{\infty}D_{n_1n_2}\psi_{\beta n_2}(q,y)=\frac{1}{\Lambda_{\alpha n_1}(p)}\int_{-1}^{1}(1-x^2)^{\alpha}f(x,y)\,\psi_{\alpha n_1}(p,x)\,dx \quad \ldots(16a)$$

$$=F_{n_1}(y),\quad \text{say} \quad \ldots(16b)$$

The right-hand side of (16a) is a sequence of functions $F_{n_1}(y)$, each represented by its single Fourier series on the left-hand side of (16a) on $[-1, 1]$, where the coefficients $D_{n_1n_2}$ of $\psi_{\beta n_2}(q, y)$ can be determined in the like manner to yield

$$D_{n_1n_2}=\frac{1}{\Lambda_{\alpha n_1}(p)\,\Lambda_{\beta n_2}(q)}\int_{-1}^{1}\int_{-1}^{1}(1-x^2)^{\alpha}(1-y^2)^{\beta}f(x,y)$$

$$\times\,\psi_{\alpha n_1}(p,x)\,\psi_{\beta n_2}(q,y)\,dx\,dy$$

$$=\frac{1}{\Lambda_{\alpha n_1}(p)\,\Lambda_{\beta n_2}(q)}\,\bar{f}(p,q)\quad[\text{from eqn. (12)}] \quad \ldots(17)$$

Thus, by virtue of (17) and (14), the inversion formula immediately follows.

3. Particular Cases

(i) Since the function $\psi_{\alpha n_1}(p, x)$ and $\psi_{\beta n_2}(q, y)$ become proportional to $T^{\alpha}_{n_1}(x)$ and $T^{\beta}_{n_2}(y)$ as p and q tend to zero, respectively, the spheroidal transform defined above reduces to the well known Gegenbauer transform.

(ii) For $p = q = 0$, and $\alpha = \beta = -\frac{1}{2}$, the transform (12) reduces to the Chebyshev transform, where the notation used by Sterling (1960) is

$$T_n(x) = \cos(n \cos^{-1} x)$$

which is equal to $n\sqrt{\pi/2}\, T_n^{-1/2}(x)$ in our case.

(iii) Moreover, for $p = q = 0$ and $\alpha = \beta = 0$, our transform reduced to the well-known Legendre transform.

4. Operational Property

Theorem 2 — If $f(x, y)$ and its first partial derivatives are bounded over the square $\{f(x, y) : -1 < x < 1, -1 < y < 1\}$, the finite spheroidal transform of differential operator

$$L^{(2)} \equiv (1 - x^2)\frac{\partial^2}{\partial x^2} + (1 - y^2)\frac{\partial^2}{\partial y^2} - 2\left[(\alpha + 1)\, x \frac{\partial}{\partial x} + (\beta + 1)\, y \frac{\partial}{\partial y}\right] - [p^2x^2 + q^2y^2] \quad \text{...(18)}$$

exists and is given by

$$T\{L^{(2)} f(x, y)\} = -[b_{\alpha n_1}(p) + b_{\beta n_2}(q)]\, T\{f(x, y)\} \quad \text{...(19)}$$

provided that

(i) $$\lim_{x \to \pm 1} (1 - x^2)^{\alpha+1} f(x, y) = \lim_{x \to \pm 1} (1 - x^2)^{\alpha+1} \frac{\partial f}{\partial x} = 0 \quad \text{...(20)}$$

(ii) $$\lim_{y \to \pm 1} (1 - y^2)^{\beta+1} f(x, y) = \lim_{y \to \pm 1} (1 - y^2)^{\beta+1} \frac{\partial f}{\partial y} = 0. \quad \text{...(21)}$$

Proof: By definition (12), we first evaluate

$$T\left\{(1 - x^2)\frac{\partial^2 f}{\partial x^2} - 2(\alpha + 1)\, x \frac{\partial f}{\partial x} - p^2x^2 f\right\}$$

$$= \int_{-1}^{1} (1 - y^2)^\beta \psi_{\beta n_2}(q, y)\left[\int_{-1}^{1} \frac{\partial}{\partial x}\left\{(1 - x^2)^{\alpha+1} \frac{\partial f}{\partial x}\right\} \psi_{\alpha n_1}(p, x)\, dx\right] dy$$

$$- \int_{-1}^{1}\int_{-1}^{1} p^2x^2(1 - x^2)^\alpha (1 - y^2)^\beta f(x, y)\, \psi_{\alpha n_1}(p, x)\, \psi_{\beta n_2}(q, y)\, dx\, dy,$$

$$= I_1 - T\{p^2x^2 f(x, y)\}, \text{ say.}$$

On evaluating the x-integral of I_1 twice by parts, the first term of I_1 vanishes on both the limits under the prescribed conditions (20) and (21). Hence

$$T\left\{(1-x^2)\frac{\partial^2 f}{\partial x^2} - 2(\alpha+1)\,x\frac{\partial f}{\partial x} - p^2x^2 f\right\}$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[\frac{d}{dx}\left\{(1-x^2)^{\alpha+1}\frac{d}{dx}\,\psi_{\alpha n_1}(p,x)\right\} - p^2x^2(1-x^2)^{\alpha}\,\psi_{\alpha n_1}(p,x)\right]$$

$$\times f(x,y)\,(1-y^2)^{\beta}\,\psi_{\beta n_2}(q,y)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1}[-b_{\alpha n_1}(p)(1-x^2)^{\alpha}\,\psi_{\alpha n_1}(p,x)]\,f(x,y)(1-y^2)^{\beta}\psi_{\beta n_2}(q,y)\,dx\,dy.$$

[from eqn. (2)]

Thus, by virtue of (12) we get

$$T\left\{(1-x^2)\frac{\partial^2 f}{\partial x^2} - 2(\alpha+1)\,x\frac{\partial f}{\partial x} - p^2x^2 f\right\} = -\,b_{\alpha n_1}(p)\,T\,\{f(x,y)\}. \qquad \text{...(22)}$$

Similarly, we can obtain

$$T\left\{(1-y^2)\frac{\partial^2 f}{\partial y^2} - 2(\beta+1)\,y\frac{\partial f}{\partial y} - q^2y^2 f\right\} = -\,b_{\beta n_2}(q)\,T\,\{f(x,y)\}. \qquad \text{...(23)}$$

Adding (22) and (23) we obtain (19).

5. Application

Formulation of the Problem

We consider the Laminar fully developed flow of an incompressible fluid inside an infinitely long channel of square cross-section at rest. We further stipulate the linearly varying temperature along the faces of the channel. The flow in considered two-dimensional and the z-axis being taken in the direction of flow. The walls are taken to be given by $x = \pm 1$, $y = \pm 1$ and all physical quantities to be the function of x, y and t. Because of the equation of conductivity we have considered the well known velocity field.

Let $\theta(x, y, t)$ be the temperature function inside the channel, $V(x, y)$ be the velocity, in the direction of z-axis.

Thus, the heat conduction equation becomes

$$\rho c_1\left[\frac{\partial\theta}{\partial t} + V(x,y)\frac{\partial\theta}{\partial z}\right] = \frac{\partial}{\partial x}\left(K_x\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial\theta}{\partial y}\right) + Q(x,y,t) \qquad \text{...(24)}$$

where $\partial\theta/\partial z = \lambda$ (constant) and the conductivities in the directions of x-axis and y-axis are given by

$$K_x = K_0(1 - x^2) \text{ and } K_y = K_0(1 - y^2)$$

respectively and the source of heat generation being given by

$$Q(x, y, t) = -K_0\left[2\alpha x \frac{\partial\theta}{\partial x} + 2\beta y \frac{\partial\theta}{\partial y} + p^2x^2 + q^2y^2\right]\theta \qquad \text{...(25)}$$

i.e. in place of heat generation there is absorption of heat. Thus, the heat conduction equation now becomes

$$\left[\frac{\partial\theta}{\partial t} + \lambda V(x, y)\right] = \frac{K_0}{\rho c_1}\left[(1 - x^2)\frac{\partial^2}{\partial x^2} + (1 - y^2)\frac{\partial^2}{\partial y^2} - 2(\alpha + 1)x\frac{\partial}{\partial x} - 2(\beta + 1)y\frac{\partial}{\partial y} - p^2x^2 - q^2y^2\right]. \qquad \text{...(26)}$$

The boundary conditions are :

$$\left.\begin{array}{ll} \text{(i)} & (1 - x^2)\dfrac{\partial\theta}{\partial x} = 0 \\ \text{(ii)} & (1 - y^2)\dfrac{\partial\theta}{\partial y} = 0 \\ \text{(iii)} & V = 0 \end{array}\right\} \text{ at } x = \pm 1 \text{ and } y = \pm 1;\ t \geqslant 0; \qquad \text{...(27)}$$

and the initial conditions are :

$$\theta = 0,\ t = 0,\ -1 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 1. \qquad \text{...(28)}$$

In this problem the velocity of the liquid is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \gamma \text{ (constant)} \qquad \text{...(29)}$$

which yields, under the prescribed boundary conditions [see Appendix A] :

$$V(x, y) = 4\gamma \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n+1}}{q_m q_n (q_m^2 + q_n^2)} \cos q_n x \cos q_m y \qquad \text{...(30)}$$

where $q_m = (2m + 1)\pi/2,\ q_n = (2n + 1)\pi/2.$

Equation (30) can be easily expressed in the form

$$V(x, y) = \frac{\gamma}{2}(1 - y^2) - 2\gamma \sum_{m=0}^{\infty} \frac{(-1)^m}{(q_m^3)} \cdot \frac{\cosh q_m x}{\cosh q_m} \cos q_m y. \qquad \text{...(31)}$$

Now, substituting this value of V in eqn. (26), we get

$$\frac{\partial\theta}{\partial t} + \lambda\left[\frac{\gamma}{2}(1-y^2) - 2\gamma\sum_{m=0}^{\infty}\frac{(-1)^m}{q_m^3}\,\frac{\cosh q_m x \cos q_m y}{\cosh q_m}\right]$$

$$= \frac{K_0}{\rho c_1}\left[(1-x^2)\frac{\partial^2}{\partial x^2} + (1-y^2)\frac{\partial^2}{\partial y^2} - 2(\alpha+1)\,x\frac{\partial}{\partial x}\right.$$

$$\left. - 2(\beta+1)\,y\frac{\partial}{\partial y} - p^2x^2 - q^2y^2\right]\theta. \qquad \ldots(32)$$

To solve this equation under the prescribed boundary conditions, we apply finite spheroidal transform (12) together with the operational property (19), and thus we obtain

$$\frac{d\bar{\theta}}{dt} + \frac{K_0}{\rho c_1}\,[b_{\alpha n_1}(p) + b_{\beta n_2}(q)]\,\bar{\theta} = -\,\lambda\gamma I(\alpha, \beta) \qquad \ldots(33)$$

where, for convenience, $I(\alpha, \beta)$ denotes the value of the double integral given by

$$I(\alpha, \beta) = -\,\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,d_0(p \mid \alpha n_1)\,d_2(q \mid \beta n_2)}{(2\alpha+1)\,(2\beta+3)\,(2\beta+5)}$$

$$-\,4\pi\sum_{m=0}^{\infty}\;\sum_{k=0}^{\infty}{}^{*}\;\sum_{k'=0}^{\infty}{}^{*}\left[\frac{d_k(p \mid \alpha n_1)\,d_{k'}(q \mid \beta n_2)}{(q_m)^{\alpha+\beta+(7/2)}k!k'!}\right.$$

$$\left.\times\,\frac{(-1)^{(k'/2)+m}\Gamma(k+2\alpha+1)\,\Gamma(k'+2\beta+1)\,I_{k+\alpha+(1/2)}(q_m)\,J_{k'+\beta+(1/2)}(q_m)}{\cosh q_m}\right], \qquad \ldots(34)$$

when n_1 and n_2 are even

$= 0$, when n_1 and n_2 are odd

and $\bar{\theta} = 0$ at $t = 0$, $\bar{\theta}$ is finite spheroidal transform (12) of $\theta(x, y, t)$. Hence

$$\bar{\theta} = \frac{-\,\lambda\gamma I(\alpha, \beta)\,\rho c_1}{K_0\,[b_{\alpha n_1}(p) + b_{\beta n_2}(q)]}\left[1 - \exp\left[-\,\frac{tK_0}{\rho c_1}\{b_{\alpha n_1}(p) + b_{\beta n_2}(q)\}\right]\right] \qquad \ldots(35)$$

where $I(\alpha, \beta)$ is given by (34) (see Appendix B). Now using the inversion formula (13) we finally get

$$\theta(x, y, t) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{\bar{\theta}\psi_{\alpha n_1}(p, x)\,\psi_{\beta n_2}(q, y)}{\Lambda_{\alpha n_1}(p)\,\Lambda_{\beta n_2}(q)}$$

where $\bar{\theta}$ is given by (35).

The finite spheroidal transform (12) can be extended to N-dimension case by considering the function of N-variables.

6. Conclusion

The problem considered in this paper can serve as a good mathematical model for considering lubricant flow between two very slow moving parts. Also, such type of problems occur in the case of oxidizing CO to CO_2 which is one of the methods followed to lessen the pollution due to exhaust gases in internal combustion engines. Moreover, the multiple finite spheroidal transform developed in this paper seems to be a much powerful tool for solving the problems governing complicated partial differential equations by converting them into simple linear differential equations.

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Appendix A

Here we solve the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \gamma \text{ (constant)} \qquad \text{...(A1)}$$

under the boundary conditions:

$$V(x, y) = 0, \quad \text{at} \quad x = \pm 1, \; y = \pm 1. \qquad \text{...(A2)}$$

Due to symmetric consideration, the flow is considered in the region $x \geqslant 0$, $y \geqslant 0$, and hence the appropriate boundary conditions (A2) now become :

$$V(x, y) = 0, \quad \text{at} \quad x = y = 0 \quad \text{and} \quad x = y = 1. \qquad \text{...(A3)}$$

To solve (A1), let us define double finite cosine transform

$$\bar{V}(m, n) = \int_0^1 \int_0^1 V(x, y) \cos q_n x \cos q_m y \, dx \, dy. \qquad \text{...(A4)}$$

where $q_n = \frac{2n+1}{2}\pi, \quad q_m = \frac{2m+1}{2}\pi.$

The inversion formula for this transform is given by

$$V(x, y) = 4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{V}(m, n) \cos q_n x \cos q_m y. \qquad \text{...(A5)}$$

Now multiplying both the sides of (A1) by $\cos q_n x \cos q_m y$ and integrating between the limits 0 to 1, we get

$$\int_0^1 \int_0^1 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \cos q_n x \cos q_m y \, dx \, dy$$

$$= \gamma \int_0^1 \int_0^1 \cos q_n x \cos q_m y \, dx \, dy \qquad \text{...(A6)}$$

or $\quad -(q_n^2 + q_m^2)\, \bar{V}(m, n) = (-1)^{m+n} \gamma / q_n q_m$

or $$\bar{V}(m, n) = \frac{(-1)^{m+n+1}\gamma}{q_n q_m (q_n^2 + q_m^2)}. \qquad \text{...(A7)}$$

Using the inversion formula (A5), we get (30).

APPENDIX B

To obtain the value of the double integral $I(\alpha, \beta)$ occurring on the right hand side of (33), we evaluate the following four interesting integrals :

(i) $$I_1 = \int_{-1}^{1} (1 - x^2)^\alpha \psi_{\alpha n_1}(p, x)\, dx, \quad \alpha > -1.$$

$$= \sum_{k=0,1}^{\infty}{}^{*} d_k(p \mid \alpha n_1) \int_{-1}^{1} (1 - x^2)^\alpha \, T_k^\alpha (x)\, dx$$

[because of the expansion (3), and term by term integration being valid due to uniform convergence of $\psi_{\alpha n_1}(p, x)$]

$$= \sum_{k=0,1}^{\infty}{}^{*} d_k(p \mid \alpha n_1) \frac{\sqrt{\pi}}{2^\alpha \Gamma(\alpha + \frac{1}{2})} \int_{-1}^{1} T_0^\alpha (x)\, T_k^\alpha (x)\, dx$$

$$= \frac{2^\alpha d_0(p \mid \alpha n_1)\, \Gamma(\alpha + 1)}{(2\alpha + 1)}.$$

[by virtue of relations (7) and (10), and $\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\,\Gamma(x + \frac{1}{2})$]

(ii) $$I_2 = \int_{-1}^{1} (1 - y^2)^{\beta+1} \psi_{\beta n_2}(q, y)\, dy.$$

Proceeding as above and making use of (9), we get

$$I_2 = \frac{-2^{\beta+2} d_2(q \mid \beta n_2)\, \Gamma(\beta + 2)}{(2\beta + 3)(2\beta + 5)}, \quad \text{when } n_2 \text{ is even}$$

$$= 0, \quad \text{when } n_2 \text{ is odd.}$$

(iii) $$I_3 = \int_{-1}^{1} (1 - x^2)^{\alpha} \cosh q_m x \psi_{\alpha n_1}(p, x)\, dx$$

$$= \tfrac{1}{2} \sum_{k=0,1}^{\infty}{}^{*} d_k(p \mid \alpha n_1) \Big[\int_{-1}^{1} (1 - x^2)^{\alpha} e^{q_m x} T_k^{\alpha}(x)\, dx$$

$$+ \int_{-1}^{1} (1 - x^2)^{\alpha} e^{-q_m x} T_k^{\alpha}(x)\, dx \Big].$$

[Because of the uniform convergence of the series, term by term integration is justified.]

Using the result (11), we have

$$I_3 = \tfrac{1}{2} \sum_{k=0,1}^{\infty}{}^{*} d_k(p \mid \alpha n_1) \frac{i^k \Gamma(k + 2\alpha + 1)}{k!}$$

$$\times \left[\frac{J_{k+\alpha+(1/2)}(-iq_m)}{(-iq_m)^{\alpha+(1/2)}} + \frac{J_{k+\alpha+(1/2)}(iq_m)}{(iq_m)^{\alpha+(1/2)}} \right]$$

$$= \tfrac{1}{2} \sum_{k=0,1}^{\infty}{}^{*} d_k(p \mid \alpha n_1) \frac{\sqrt{2\pi}}{k!} \Gamma(k + 2\alpha + 1) [1 + (-1)^k] \frac{I_{k+\alpha+(1/2)}(q_m)}{(q_m)^{\alpha+(1/2)}}$$

$$\left[\because \quad J_n(-it) = (-1)^n J_n(it),\ I_n(t) = i^{-n} J_n(it) = \sum_{s=0}^{\infty} \frac{(t/2)^{n+2s}}{\Pi(s)\, \Pi(n + s)} \right]$$

Hence

$$I_3 = \frac{\sqrt{2\pi}}{q^{\alpha+(1/2)}} \sum_{k=0}^{\infty}{}^{*} d_k(p \mid \alpha n_1) \frac{\Gamma(k + 2\alpha + 1)}{k!} I_{k+\alpha+(1/2)}(q_m),$$

when n_1 is even

$$= 0, \quad \text{when } n_1 \text{ is odd.}$$

(iv) $$I_4 = \int_{-1}^{1} \cos q_m y (1 - y^2)^{\beta} \, \psi_{\beta n_2}(q, y) \, dy$$

$$= \text{Real part of } \int_{-1}^{1} (1 - y^2)^{\beta} \, e^{i q_m y} \psi_{\beta n_2}(q, y) \, dy$$

$$= \frac{\sqrt{2\pi}}{q_m^{\beta+1}} \sum_{k'=0}^{\infty}{}^{*} \frac{d_{k'}(q \mid \beta n_2) \, \Gamma(k' + 2\beta + 1) \, (-1)^{k'/2}}{k'!} J_{k'+\beta+(1/2)}(q_m),$$

when n_2 is even,

$= 0$, when n_2 is odd.

By substituting the values of I_1, I_2, I_3, I_4 integrals in the integral $I(\alpha, \beta)$, the result (34) immediately follows.