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AN APPELL CROSS-SEQUENCE SUGGESTED BY THE BERNOULLI AND EULER POLYNOMIALS OF GENERAL ORDER

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The Appell cross-sequence $\{R_n^{(\alpha)}(x; a, r, b)\}$ relative to the invertible shift invariant operator $\left(\frac{\Delta + bI}{aD^r}\right)^\alpha$, where a, b are real numbers ($a \neq 0$), α is any integer and r is a non-negative integer, is studied by the use of finite operator calculus developed by Rota *et al.* (1973). This sequence unifies the sequences of Bernoulli and Euler polynomials of general order and their numerous special cases. We give several expansions, addition formulae, binomial identity, recurrence relations, generating relations, finite difference formulae and several other results for the polynomials $R_n^{(\alpha)}(x; a, r, b)$. On specialization of parameters one gets well-known properties of the Bernoulli and the Euler polynomials of order α, and indeed those of Bernoulli and Euler polynomials, as well as of Bernoulli and Euler numbers. We also get results on several not so well-known extensions of these polynomials discussed in recent years.

1. INTRODUCTION

It is well-known that Bernoulli numbers and Bernoulli polynomials are of basic importance in several parts of analysis and calculus of finite differences and have applications in various fields such as Statistics, Numerical analysis, Combinatorics, etc. They have been discussed in details in various standard treatises (see Erdélyi 1953, Rainville 1960, Whittaker and Watson 1962, Hildebrand 1974, Brand 1966, Milne-Thomson 1951, Nörlund 1954, Johnson and Kotz 1969). Another polynomial set which is related to Bernoulli polynomials and also has interesting properties is the set of Euler polynomials (see Erdélyi 1953, Rainville 1960, Milne-Thomson 1951, Nörlund 1954, Johnson and Kotz 1969). Various extensions of both Bernoulli and Euler polynomials have been discussed from time to time by a number of authors (see Erdélyi 1953, 1955, Karande and Thakare 1975, Nörlund 1954, Milne-Thomson 1951).

In this paper we systematically study an Appell cross-sequence of polynomials which is of a very general nature and subsumes the Bernoulli and Euler polynomials of order α, the Bernoulli and Euler polynomials, as also several other related polynomials. Our discussion is based on the techniques of finite operator calculus developed recently by Rota *et al.* (1973). It may be remarked that this study by Rota *et al.* is the first rigorous and systematic version of finite operator calculus and symbolic calculus considered by numerous mathematicians during the last several decades.

It is known (Rota *et al.* 1973) that the Bernoulli polynomials of order α form an Appell set relative to the invertible shift invariant operator J^{α}, where

$$Jp(x) = \int_{x}^{x+1} p(t)\, dt$$

for all polynomials $p(x)$ in the algebra P of polynomials over a field of characteristic zero. In this paper, we study a much more general polynomial set $R_n^{(\alpha)}(x; a, r, b)$ which is an Appell cross-sequence relative to the invertible shift invariant operator

$$S_{\alpha,r,a,b} = \left(\frac{\Delta + bI}{aD^r}\right)^{\alpha}.$$

For special values of α, r, a and b, we get the following numerous polynomial sets :

1. Bernoulli polynomials

$$R_n^{(1)}(x; 1, 1, 0) = B_n(x)$$

2. Bernoulli polynomials of order α

$$R_n^{(\alpha)}(x; 1, 1, 0) = B_n^{(\alpha)}(x)$$

3. Bernoulli numbers

$$R_n^{(1)}(0; 1, 1, 0) = B_n$$

4. Bernoulli numbers of order α

$$R_n^{(\alpha)}(0; 1, 1, 0) = B_n^{(\alpha)}$$

5. Euler polynomials

$$R_n^{(1)}(x; 2, 0, 2) = E_n(x)$$

6. Euler polynomials of order α

$$R_n^{(\alpha)}(x; 2, 0, 2) = E_n^{(\alpha)}(x)$$

7. Euler numbers

$$2^n R_n^{(1)}(\tfrac{1}{2}; 2, 0, 2) = E_n$$

8. Euler numbers of order α

$$2^n R_n^{(\alpha)}\left(\frac{\alpha}{2};\, 2, 0, 2\right) = E_n^{(\alpha)}$$

9. Eulerian polynomials (see Carlitz 1962)

$$R_n^{(1)}\left(x;\, 1 - \frac{1}{\xi}, 0, 1 - \frac{1}{\xi}\right) = \Phi_n(x; \xi), \quad \text{where} \quad \xi \neq 1$$

10. Genocchi polynomials

$$R_n^{(1)}(x; 2, 1, 2) = G_n(x)$$

11. Genocchi numbers

$$R_n^{(1)}(0; 2, 1, 2) = G_n$$

12. Polynomial $D_n(x; a, k)$ (see Karande and Thakare 1975)

$$R_n^{(1)}\left(x; \frac{1}{2^{k-1}}, k, 1 - a\right) = D_n(x; a, k).$$

Throughout, we will use the notations and terminology of Rota *et al.* (1973).

2. Polynomial $R_n^{(\alpha)}(x; a, r, b)$ as a Sheffer Set

We define $R_n^{(\alpha)}(x; a, r, b)$ to be the Appell set relative to the invertible shift invariant operator

$$S_{\alpha, r, a, b} = \left(\frac{\Delta + bI}{aD^r}\right)^\alpha \qquad \text{...(2.1)}$$

where a, b are real numbers and $a \neq 0$; also α is any integer (positive, negative or zero) and r is a non-negative integer. As usual D is the differential operator d/dx and Δ is the difference operator. Evidently, the delta operator for our Appell sequence is D and the basic sequence is $\{x^n\}$. Since $R_n^{(\alpha)}(x; a, r, b)$ is a Sheffer set relative to S, we have (Rota *et al.* 1973, §5, Prop. 1)

$$R_n^{(\alpha)}(x; a, r, b) = S_{\alpha, r, a, b}^{-1}\, x^n$$

$$= \left(\frac{aD^r}{\Delta + bI}\right)^\alpha x^n. \qquad \text{...(2.2)}$$

Indeed

$$DR_n^{(\alpha)}(x; a, r, b) = nR_{n-1}^{(\alpha)}(x; a, r, b) \qquad \text{...(2.3)}$$

which immediately gives

$$D^p R_n^{(\alpha)}(x; a, r, b) = (n)_p R_{n-p}^{(\alpha)}(x; a, r, b) \qquad \text{...(2.4)}$$

where $\quad (n)_p = n(n-1) \ldots (n-p+1).$

The binomial identity for the polynomials $R_n^{(\alpha)}(x; a, r, b)$ is (Rota *et al.* 1973, §5, Prop. 2)

$$R_n^{(\alpha)}(x+y; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{(\alpha)}(x; a, r, b)\, y^{n-p}. \qquad \text{...(2.5)}$$

Substitution $y = 1$ in (2.5) yields

$$R_n^{(\alpha)}(x+1; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{(\alpha)}(x; a, r, b). \qquad \text{...(2.6)}$$

Interchanging x and y in (2.5) and then substituting $y = 0$, we get

$$R_n^{(\alpha)}(x; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{(\alpha)}(0; a, r, b)\, x^{n-p}. \qquad \text{...(2.7)}$$

As $R_n^{(\alpha)}(x; a, r, b)$ is Sheffer set, its generating relation is given by (Rota *et al.* 1973, §5, Prop. 5)

$$\sum_{n \geqslant 0} R_n^{\alpha}(x; a, r, b) \frac{t^n}{n!} = \frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} \qquad \text{...(2.8)}$$

where $s(t) = \left(\dfrac{e^t + b - 1}{at^r}\right)^{\alpha}$ is the indicator of $S_{\alpha,r,a,b}$, and $q^{-1}(t)$ is the formal power series inverse to $q(t)$, the indicator of the delta operator. Evidently, here $q(t) = t$, so that from (2.8) we get the following generating relation for $R_n^{(\alpha)}(x; a, r, b)$:

$$\sum_{n \geqslant 0} R_n^{(\alpha)}(x; a, r, b) \frac{t^n}{n!} = \left(\frac{at^r}{e^t + b - 1}\right)^{\alpha} e^{xt}. \qquad \text{...(2.9)}$$

Now from the second expansion theorem (Rota *et al.* 1973, §5, Th. 6) we have that if T is any shift invariant operator and $f(x)$ a polynomial, then

$$Tf(x+y) = \sum_{n \geqslant 0} \frac{R_n^{(\alpha)}(y; a, r, b)}{n!} D^n \left(\frac{\Delta + bI}{aD^r}\right)^{\alpha} Tf(x). \qquad \text{...(2.10)}$$

In particular, if $T = D^p$ we have from (2.10)

$$D^p f(x + y) = \sum_{n \geqslant 0} \frac{R_n^{(\alpha)}(y; a, r, b)}{n!} \left(\frac{\Delta + bI}{a}\right)^{\alpha} f^{n+p-r\alpha}(x). \qquad \ldots(2.11)$$

If $p = 1$, $\alpha = 1$, $a = 1$, $r = 1$, $b = 0$, (2.11) reduces to the well-known Euler-Maclaurin theorem [Milne-Thomson 1951, §6.511(3)]

$$Df(x + y) = \sum_{n \geqslant 0} \frac{B_n(y)}{n!} \Delta f^{(n)}(y).$$

3. Some More Results on $R_n^{(\alpha)}(x; a, r, b)$

From (2.9), we have

$$\sum_{n \geqslant 0} R_n^{(\alpha)}(\alpha - x; a, r, b) \frac{t^n}{n!} = \left(\frac{at^r}{e^t + b - 1}\right)^{\alpha} e^{(\alpha - x)t}$$

$$= \frac{(-1)^{r\alpha}}{(b-1)^{\alpha}} \left(\frac{a(-t)^r}{e^{-t} + \frac{b}{b-1} - 1}\right)^{\alpha} e^{x(-t)}$$

$$= \frac{(-1)^{r\alpha}}{(b-1)^{\alpha}} \sum_{n \geqslant 0} (-1)^n R_n^{(\alpha)}\left(x; a, r, \frac{b}{b-1}\right) \frac{t^n}{n!}.$$

Equating coefficients of t^n on both sides, we get

$$R_n^{(\alpha)}(\alpha - x; a, r, b) = \frac{(-1)^{r\alpha + n}}{(b-1)^{\alpha}} R_n^{(\alpha)}\left(x; a, r, \frac{b}{b-1}\right). \qquad \ldots(3.1)$$

Again from (2.9), we have

$$\sum_{n \geqslant 0} [R_n^{(\alpha)}(x + 1; a, r, b) + (b - 1) R_n^{(\alpha)}(x; a, r, b)] \frac{t^n}{n!}$$

$$= at^r \left(\frac{at^r}{e^t + b - 1}\right)^{\alpha - 1} e^{xt}$$

$$= a \sum_{n \geqslant 0} R_n^{(\alpha - 1)}(x; a, r, b) \frac{t^{n+r}}{n!}$$

$$= a \sum_{n=r}^{\infty} R_{n-r}^{(\alpha - 1)}(x; a, r, b) \frac{t^n}{(n-r)!}$$

which gives

$$R_n^{(\alpha)}(x+1; a, r, b) + (b-1) R_n^{(\alpha)}(x; a, r, b)$$

$$= a(n)_r R_{n-r}^{(\alpha-1)}(x; a, r, b) \quad \text{for} \quad n \geqslant r. \qquad \ldots(3.2)$$

Replacing x by $\alpha - x - 1$ in (3.2), we get

$$R_n^{(\alpha)}(\alpha - x; a, r, b) = (1-b) R_n^{(\alpha)}(\alpha - x - 1; a, r, b) + a(n)_r R_{n-r}^{(\alpha-1)}(\alpha - x - 1; a, r, b),$$

$$\frac{(-1)^{r\alpha+n}}{(b-1)^\alpha} R_n^{(\alpha)}\left(x; a, r, \frac{b}{b-1}\right)$$

$$= (1-b) R_n^{(\alpha)}(\alpha - x - 1; a, r, b)$$

$$+ a(n)_r R_n^{(\alpha-1)}(\alpha - x - 1; a, r, b) \text{ (using (3.1))}$$

$$\frac{(-1)^{r\alpha+n+\alpha}}{(1-b)^\alpha} R_n^{(\alpha)}\left(x; a, r, \frac{b}{b-1}\right)$$

$$= (1-b) R_n^{(\alpha)}(\alpha - x - 1; a, r, b)$$

$$+ a(n)_r R_{n-r}^{(\alpha-1)}(\alpha - x - 1; a, r, b). \qquad \ldots(3.3)$$

Again from (2.9), we have

$$\sum_{n \geqslant 0} R_n^{(\alpha)}(x; a, r, b(2-b)) \frac{t^n}{n!} = \left(\frac{at^r}{e^t - (1-b)^2}\right)^\alpha e^{xt}$$

which by replacement of t by $2t$ reduces to

$$\sum_{n \geqslant 0} R_n^{(\alpha)}(x; a, r, b(2-b)) \frac{(2t)^n}{n!} = \left(\frac{a(2t)^n}{e^{2t} - (1-b)^2}\right)^\alpha e^{2xt}$$

or, $$\sum_{n \geqslant 0} 2^n R_n^{(\alpha)}(x; a, r, b(2-b)) \frac{t^n}{n!}$$

$$= 2^{r\alpha-\alpha}\left(\frac{at}{e^t + b - 1}\right)^\alpha \left(\frac{2t^{r-1}}{e^t + 1 - b}\right)^\alpha e^{2xt}$$

$$= 2^{\alpha(r-1)} \sum_{n=0}^{\infty} \sum_{p=0}^{n} R_{n-p}^{(\alpha)}(0; a, 1, b) R_p^{(\alpha)}(2x; 2, r-1, 2-b) \frac{t^n}{p!(n+p)!}$$

which gives

$$R_n^{(\alpha)}(x; a, r, b(2-b)) = 2^{\alpha(r-1)-n} \sum_{p=0}^{n} \binom{n}{p} R_{n-p}^{(\alpha)}(0; a, 1, b) \times R_p^{(\alpha)}(2x; 2, r-1, 2-b) \qquad \ldots(3.4)$$

Also from (2.9),

$$\sum_{n\geqslant 0} \left[R_n^{(\alpha)}\left(\frac{x+1}{2}; a, r, b(2-b)\right) - (1-b) R_n^{(\alpha)}\left(\frac{x}{2}, a, r, b(2-b)\right)\right] \frac{t^n}{n!}$$

$$= \left(\frac{at^r}{e^t-(1-b)^2}\right)^{\alpha} e^{(x+1/2)t} - (1-b)\left(\frac{at^r}{e^t-(1-b)^2}\right)^{\alpha} e^{xt/2}$$

$$= 2^{r\alpha-2\alpha} t \left(\frac{2a\left(\frac{t}{2}\right)^{r-1}}{e^{t/2}+(2-b)-1}\right)^{\alpha} e^{(x/2)t} \left(\frac{2\left(\frac{t}{2}\right)}{e^{t/2}+b-1}\right)^{\alpha-1}$$

$$= 2^{r\alpha-2\alpha} \sum_{n\geqslant 0}\sum_{p=0}^{n-1} R_p^{(\alpha)}(x; 2a, r-1, 2-b) R_{n-p-1}^{(\alpha-1)}(0; 2, 1, b) \times \frac{t^n}{2^{n-1} p!(n-1-p)!}.$$

Equating coefficients of t^n on both sides, we get

$$R_n^{(\alpha)}\left(\frac{x+1}{2}; a, r, b(2-b)\right) - (1-b) R_n^{(\alpha)}\left(\frac{x}{2}; a, r, b(2-b)\right)$$

$$= 2^{r\alpha-2\alpha} \sum_{p=0}^{n-1} R_p^{(\alpha)}(x; 2a, r-1, 2-b) R_{n-p-1}^{(\alpha-1)}(0; 2, 1, b) \times \frac{n!}{2^{n-1} p!(n-p-1)!}. \qquad \ldots(3.5)$$

Applying operator $(\triangle + bI)$ on (2.2), we have

$$(\triangle + bI) R_n^{(\alpha)}(x; a, r, b) = (\triangle + bI)\left(\frac{aD^r}{\triangle + bI}\right)^{\alpha} x^n$$

$$= a(n)_r R_{n-r}^{(\alpha-1)}(x; a, r, b). \qquad \ldots(3.6)$$

Also from (2.2), we find that

$$(\triangle + bI)^{\alpha} R_n^{(\alpha)}(x; a, r, b) = (\triangle + bI)^{\alpha}\left(\frac{aD^r}{\triangle + bI}\right)^{\alpha} x^n$$

$$= a^{\alpha}(n)_{r\alpha} x^{n-r\alpha} \qquad \ldots(3.7)$$

and

$$S_{\alpha,r,a,b}R_n^{(\alpha)}(x;\, a,\, r,\, b) = x^n. \qquad \ldots(3.8)$$

4. A Recurrence Relation for $R_n^{(\alpha)}(x;\, a,\, r,\, b)$

In this section, we derive a recurrence relation for $R_n^{(\alpha)}(x; a, r, b)$ using Pincherle derivative defined on the algebra of all shift invariant operators. Recall (Rota *et al.* 1973, §4) that Pincherle derivative of an operator T is defined as $T' = TX - XT$, where X is the multiplication operator defined on P by $X : p(x) \to xp(x)$.

We begin with

$$\left(\left(\frac{aD^r}{\triangle + bI}\right)^\alpha\right)' f(x) = \left(\left(\frac{aD^r}{\triangle + bI}\right)^\alpha X - X\left(\frac{aD^r}{\triangle + bI}\right)^\alpha\right) f(x)$$

and get

$$\begin{aligned}\left(\frac{aD^r}{\triangle + bI}\right)^\alpha xf(x) &= \left(\left(\frac{aD^r}{\triangle + bI}\right)^\alpha\right)' f(x) + x\left(\frac{aD^r}{\triangle + bI}\right)^\alpha f(x)\\ &= \frac{\alpha r a^r D^{r\alpha-1}}{(\triangle + bI)^\alpha} f(x) - \alpha\left(\frac{aD^r}{\triangle + bI}\right)^\alpha f(x) + (b-1)\,\alpha\,\frac{(aD^r)^\alpha}{(\triangle + bI)^{\alpha+1}} f(x)\\ &\quad + x\left(\frac{aD^r}{\triangle + bI}\right)^\alpha f(x).\end{aligned}$$

If $f(x) = x^{n-1}$, then we have

$$\begin{aligned}\left(\frac{aD^r}{\triangle + bI}\right)^\alpha x^n &= \frac{\alpha r a^r D^{r\alpha-1}}{(\triangle + bI)^\alpha} x^{n-1} + (x-\alpha)\left(\frac{aD^r}{\triangle + bI}\right)^\alpha x^{n-1}\\ &\quad + (b-1)\,\alpha\,\frac{(aD^r)^\alpha}{(\triangle + bI)^{\alpha-1}} x^{n-1}\\ &= \frac{\alpha r}{n}\left(\frac{aD^r}{\triangle + bI}\right)^\alpha x^n + (x-\alpha)\left(\frac{aD^r}{\triangle + bI}\right)^\alpha x^{n-1}\\ &\quad + \frac{\alpha(b-1)}{a}\left(\frac{aD^r}{\triangle + bI}\right)^{\alpha+1}\frac{(n-1)!}{(n+r-1)!}x^{n+r-1}\end{aligned}$$

which leads to a recurrence relation

$$\begin{aligned}&\left(1 - \frac{\alpha r}{n}\right) R_n^{(\alpha)}(x;\, a,\, r,\, b) + (\alpha - x)\, R_{n-1}^{(\alpha)}(x;\, a,\, r,\, b)\\ &\quad = \frac{\alpha(b-1)}{a}\,\frac{(n-1)!}{(n+r-1)!}\, R_{n+r-1}^{(\alpha+1)}(x;\, a,\, r,\, b).\end{aligned} \qquad \ldots(4.1)$$

Substitution $\alpha = n$ in (4.1) yields

$$\begin{aligned}&(1 - r)\, R_n^{(n)}(x;\, a,\, r,\, b) + (n - x)\, R_{n-1}^{(n)}(x;\, a,\, r,\, b)\\ &\quad = \frac{n!}{(n+r-1)!}\,\frac{b-1}{a}\, R_{n+r-1}^{(n+1)}(x;\, a,\, r,\, b).\end{aligned} \qquad \ldots(4.2)$$

5. $R_n^{(\alpha)}(x; a, r, b)$ AS A CROSS-SEQUENCE

It is easily seen that $P^{-\alpha} = \left(\frac{aD^r}{\Delta + bI}\right)^\alpha$ form a one parameter group of shift invariant operators and for the sequence $q_n(x) = x^n$ (of binomial type), the relation

$$R_n^{[\alpha]}(x; a, r, b) = P^{-\alpha}x^n \quad \text{...(5.1)}$$

holds and hence $R_n^{[\alpha]}(x; a, r, b)$ is a cross-sequence (Rota *et al.* 1973, §8, Th. 8). Evidently

$$R_n^{[\alpha]}(x; a, r, b) = R_n^{(\alpha)}(x; a, r, b). \quad \text{...(5.2)}$$

Since $R_n^{[\alpha]}(x; a, r, b)$ is a cross-sequence, the identity

$$R_n^{[\alpha+\beta]}(x + y; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{[\alpha]}(x; a, r, b)\, R_{n-p}^{[\beta]}(y; a, r, b),\ \forall n, \quad \text{...(5.3)}$$

holds (Rota *et al.* 1973) for all α and β and for any x and y.

Putting $\beta = -\alpha$ (5.3), we have

$$(x + y)^n = \sum_{p=0}^{n} \binom{n}{p} R_p^{[\alpha]}(x; a, r, b)\, R_{n-p}^{[-\alpha]}(y; a, r, b) \quad \text{...(5.4)}$$

when $y = 0$, (5.4) gives

$$x^n = \sum_{p=0}^{n} \binom{n}{p} R_p^{[\alpha]}(x; a, r, b)\, R_{n-p}^{[-\alpha]}(0; a, r, b). \quad \text{...(5.5)}$$

Substituting $\alpha = 1$, $\beta = \alpha - 1$ in (5.3), we get

$$R_n^{[\alpha]}(x + y; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{[1]}(x; a, r, b)\, R_{n-p}^{[\alpha-1]}(y; a, r, b). \quad \text{...(5.6)}$$

Now interchanging x and y in (5.6) and substituting $y = 0$, we get

$$R_n^{[\alpha]}(x; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{[1]}(0; a, r, b)\, R_{n-p}^{[\alpha-1]}(x; a, r, b) \quad \text{...(5.7)}$$

Putting $y = 0$ and $\beta = \gamma + 1$, we have,

$$R_n^{[\alpha+\gamma+1]}(x; a, r, b) = \sum_{p=0}^{n} \binom{n}{p} R_p^{[\alpha]}(x; a, r, b)\, R_{n-p}^{[\gamma+1]}(0; a, r, b) \qquad \ldots(5.8)$$

6. Special Cases

Putting $a = 1$, $r = 1$, $b = 0$ in the relations (2.2), (2.4), (2.5), (2.6), (2.7), (2.9), (3.1), (3.2), (3.3), (3.4), (3.6), (3.7), (3.8), (4.1), (4.2), (5.3), (5.4), (5.5), (5.6), (5.7) (5.8) we give below a numbar of results for the Bernoulli polynomials of order α; most of these are well-known. On further putting $\alpha = 1$ we get the standard results for the Bernoulli polynomials $B_n(x)$. The substitution $x = 0$ gives the well-known results on the Bernoulli numbers and also on Bernoulli numbers of order α :

(i) $B_n^{(\alpha)}(x) = \left(\dfrac{D}{\Delta}\right)^{\alpha} x^n$

(ii) $\displaystyle\sum_{n\geqslant 0} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt}$ [Erdélyi 1955, §19.7(30)]

(iii) $D^r B_n^{(\alpha)}(x) = (n)_p B_{n-p}^{(\alpha)}(x),\ p = 1, 2, \ldots$

(iv) $\displaystyle B_n^{(\alpha)}(x + y) = \sum_{p=0}^{n}\binom{n}{p} B_p^{(\alpha)}(x)\, y^{n-p}$

(v) $\displaystyle B_n^{(\alpha)}(x + 1) = \sum_{p=0}^{n}\binom{n}{p} B_p^{(\alpha)}(x)$

(vi) $\displaystyle B_n^{(\alpha)}(x) = \sum_{p=0}^{n}\binom{n}{p} B_p^{(\alpha)}\, x^{n-p}$

(vii) $B_n^{(\alpha)}(\alpha - x) = (-1)^n\, B_n^{(\alpha)}(x)$ [Milne-Thomson 1951, §6.2(1)]

which for $x = 0$, $n = 2k$, gives

$B_{2k}^{(\alpha)}(\alpha) = B_{2k}^{(\alpha)}$ [Milne-Thomson 1951, §6.2(2)]

and for $x = \frac{1}{2}\alpha$, $n = 2k + 1$, reduces to $B_{2k+1}^{(\alpha)}(\frac{1}{2}\alpha) = 0$

[Milne-Thomson 1951, §6.2(3)]

(viii) $B_n^{(\alpha)}(x+1) = B_n^{(\alpha)}(x) + nB_{n-1}^{(\alpha-1)}(x),\ n \geqslant 1$

[Milne-Thomson 1951, §6.11(7)]

(ix) $(-1)^n B_n^{\alpha}(x) = B_n^{(\alpha)}(\alpha - x - 1) + nB_{n-1}^{(\alpha-1)}(\alpha - x - 1)$

(x) $B_n^{(\alpha)}(x) = \frac{1}{2^n} \sum_{p=0}^{n} \binom{n}{p} B_{n-p}^{(\alpha)} E_p^{(\alpha)}(2x)$

(xi) $\triangle B_n^{(\alpha)}(x) = nB_{n-1}^{(\alpha-1)}(x)$

(xii) $\triangle^{\alpha} B_n^{(\alpha)}(x) = (n)_{\alpha} x^{n-\alpha}$ [Milne-Thomson 1951, §6.11(6)]

(xiii) $\left(\frac{\triangle}{D}\right)^{\alpha} B_n^{(\alpha)}(x) = J^{\alpha} B_n^{(\alpha)}(x) = x^n$

(xiv) $B_n^{(\alpha+1)}(x) = \left(1 - \frac{n}{\alpha}\right) B_n^{(\alpha)}(x) + n\left(\frac{x}{\alpha} - 1\right) B_{n-1}^{(\alpha)}(x)$

[Milne-Thomson 1951, §6.3(2)]

(xv) $B_n^{(n+1)}(x) = (x-n)\, B_{n-1}^{(n)}(x)$

which gives

$B_n^{n+1}(x) = (x-1)_n\, B_0^{(1)}(x) = (x-1)_n$ [Milne-Thomson 1951, §6.4(1)]

(xvi) $B_n^{(\alpha+\beta)}(x+y) = \sum_{p=0}^{n} \binom{n}{p} B_p^{(\alpha)}(x)\, B_{n-p}^{(\beta)}(y)$

(xvii) $(x+y)^n = \sum_{p=0}^{n} \binom{n}{p} B_p^{(\alpha)}(x)\, B_{n-p}^{(-\alpha)}(y)$ [Nörlund 1954, §74(72)]

(xviii) $x^n = \sum_{p=0}^{n} \binom{n}{p} B_{n-p}^{(\alpha)}(x)\, B_p^{(-\alpha)}$ [Nörlund 1954, §74)]

(xix) $B_n^{(\alpha)}(x+y) = \sum_{p=0}^{n} \binom{n}{p} B_p(x)\, B_{n-p}^{(\alpha-1)}(y)$

(xx) $B_n^{(\alpha)}(x) = \sum_{p=0}^{n} \binom{n}{p} B_p B_{n-p}^{(\alpha-1)}(x)$

[Milne-Thomson 1951, Q 1(iii), p. 150]

$$\text{(xxi)} \quad B_n^{(\alpha+\gamma+1)}(x) = \sum_{p=0}^{n} \binom{n}{p} B_p^{(\alpha)}(x)\, B_{n-p}^{(\gamma+1)}.$$

We can also obtain similar results for all other polynomials given in §1, notably Euler polynomials of order α, if we similarly specialize the parameters of $R_n^{(\alpha)}(x; a, r, b)$.

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