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ON $(\bar{N}, p_n)$ SUMMABILITY OF FOURIER SERIES AND ITS CONJUGATE SERIES

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In the present paper, two theorems on $(\bar{N}, p_n)$ summability have been proved which generalize the results of Luan (1973) on (l) summability.

§1. Let $\Sigma\, u_n$ be a given series with the sequence of partial sums $\{S_n\}$ and $\{p_n\}$ be a sequence of constants, real or complex, such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \neq 0 \text{ for all } n$$

and

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu S_\nu .$$

If $\lim_{n \to \infty} t_n = s$, the series $\Sigma\, u_n$ or the sequence $\{s_n\}$ is said to be summable $(\bar{N}, p_n)$ to s (Hardy 1949, p. 57). When $p_n = 1/(n+1)$, the method $(\bar{N}, p_n)$ is equivalent to the logarithmic method (l) (see Hardy 1949, p. 59 and p. 87).

§2. Let $f(t)$ be a 2π-periodic and L-integrable function over $(-\pi, \pi)$. The Fourier series of $f(t)$ and its conjugate series are given by

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad \ldots(2.1)$$

and

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t) \qquad \ldots(2.2)$$

respectively.

We write

$$\phi(t) = \tfrac{1}{2} \{f(x+t) + f(x-t) - 2S\}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}$$

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\,du$$

and

$$\bar{f}(x) = \lim_{\varepsilon\to 0} \bar{f}(x, \epsilon) = \lim_{\varepsilon\to 0}\left\{\frac{1}{\pi}\int_\varepsilon^\pi \frac{\psi(t)}{\tan t/2}\,dt\right\}.$$

§3. In this paper, we prove the following:

Theorem 1 — If

$$\int_t^\pi \frac{|\phi_1(u)|}{u}\,du = o\left(\log\frac{1}{t}\right) \quad \text{as } t\to 0 \qquad \ldots(3.1)$$

then the Fourier series (2.1) at $t = x$ is summable $(\bar{N}, p_n)$ to S, where $\{p_n\}$ is a sequence of positive real constants such that $P_n = \sum_{\nu=0}^{n} p_\nu$ tends to infinity with n and

$$\sum_{\nu=1}^{n-1} |\triangle(\nu p_\nu)| + np_n = O\left(\frac{P_n}{\log n}\right). \qquad \ldots(3.2)$$

Theorem 2 — If

$$\Psi(t) = \int_t^\pi \frac{\psi(u)}{u}\log\frac{1}{u}\,du = o\left(\log\frac{1}{t}\right) \quad \text{as } t\to 0 \qquad \ldots(3.3)$$

then

$$\bar{t}_n(x) - \bar{f}(x, (1/n)) = o(1) \text{ as } n\to\infty$$

where $\bar{t}_n(x)$ is the $(\bar{N}, p_n)$ mean of the conjugate series (2.2) at $t = x$ and $\{p_n\}$ is a sequence of constants as defined in Theorem 1 satisfying (3.2), provided that $\bar{f}(x)$ exists.

For $p_\nu = 1/(\nu+1)$, we obtain two theorems due to Luan (1973) on (l) summability.

§4. We require the following lemmas:

Lemma 1 —

(i) $$\sum_{\nu=1}^{n} \sin \nu t = \frac{\cos \frac{1}{2} t - \cos (n+\frac{1}{2}) t}{2 \sin \frac{1}{2} t} = \begin{cases} O(n) \text{ for } 0 < t \leqslant 1/n, \\ O(1/t) \text{ for } 1/n < t \leqslant \pi. \end{cases}$$

(ii) $$\sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu t} = O(n) \text{ for } 0 < t < \frac{1}{n}.$$

(iii) $$\sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = O(1) \text{ for all values of } n \text{ and } t.$$

(iv) $$\sum_{\nu=1}^{n} \frac{\sin^2 \nu t}{\nu t} = O(n^2 t) \text{ for } 0 < t \leqslant \frac{1}{n}.$$

Lemma 2 —

(i) $$\sum_{\nu=1}^{n} \cos \nu t = \frac{\sin (n+\frac{1}{2}) t - \sin \frac{1}{2} t}{2 \sin \frac{1}{2} t} = \begin{cases} O(n) \text{ for } 0 < t \leqslant 1/n, \\ O(1/t) \text{ for } 1/n < t \leqslant \pi. \end{cases}$$

(ii) $$\sum_{\nu=1}^{n} \frac{\cos \nu t}{\nu} = O\left(\log \frac{1}{t}\right) \text{ for } 0 < t < \pi.$$

The third estimate of Lemma 1 and the second estimate of Lemma 2 are well known (Titchmarsh 1952, p. 440, Hardy and Rogosinski 1947). The proofs of the remaining estimates are immediate.

Lemma 3 — Let $\{p_n\}$ be a sequence as given in Theorem 1 satisfying (3.2.) Then we have

(i) $$\sum_{\nu=0}^{n} \nu p_\nu \sin \nu t = \begin{cases} O(nP_n/\log n) \text{ for } 0 < t \leqslant 1/n, \\ O(P_n/t \log n) \text{ for } 1/n < t \leqslant \pi. \end{cases}$$

(ii) $$\sum_{\nu=0}^{n} p_\nu \sin \nu t = O(P_n/\log n) \text{ for all values of } n \text{ and } t.$$

(iii) $$\sum_{\nu=0}^{n} p_\nu \frac{\sin \nu t}{t} = O(nP_n/\log n) \text{ for } 0 < t \leqslant 1/n.$$

(iv) $$\sum_{\nu=0}^{n} p_\nu \sin^2 \frac{\nu t}{2} = O(P_n) \text{ for } t = 1/n.$$

(v) $$\sum_{\nu=0}^{n} p_\nu \frac{\sin^2 \frac{1}{2}\nu t}{t} = O(P_n n^2 t/\log n) \text{ for } 0 < t \leqslant 1/n.$$

PROOF : By Abel's transformation, we have

$$\left|\sum_{\nu=0}^{n} \nu p_\nu \sin \nu t\right| \leqslant \sum_{\nu=1}^{n-1} \left|\sum_{k=1}^{\nu} \sin kt\right| \left|\triangle (\nu p_\nu)\right| + \left|\sum_{k=1}^{n} \sin kt\right| np_n.$$

On using (3.2) and Lemma 1(i), we get estimate (i).

Similarly, applying Abel's transformation, the estimates (ii), (iii) and (v) can be easily obtained with the help of (3.2) and Lemmas 1(iii), 1(ii) and 1(iv) respectively.

The proof of estimate (iv) is immediate.

Lemma 4 — Let $\{p_n\}$ be a sequence as given in Theorem 1 satisfying (3.2). Then we have

(i) $$\sum_{\nu=0}^{n} \nu p_\nu \cos \nu t = \begin{cases} O(nP_n/\log n) & \text{for } 0 < t \leqslant 1/n, \\ O(P_n/t \log n) & \text{for } 1/n < t \leqslant \pi. \end{cases}$$

(ii) $$\sum_{\nu=0}^{n} p_\nu \cos \nu t = \begin{cases} O((P_n/\log n) \log 1/t) & \text{for } 0 < t \leqslant \pi, \\ O(P_n) & \text{for } t = 1/n. \end{cases}$$

PROOF : By Abel's transformation, we have

$$\left|\sum_{\nu=0}^{n} \nu p_\nu \cos \nu t\right| \leqslant \sum_{\nu=1}^{n-1} \left|\sum_{k=1}^{\nu} \cos kt\right| \left|\triangle (\nu p_\nu)\right| + \left|\sum_{k=1}^{n} \cos kt\right| np_n.$$

Using (3.2) and Lemma 2(i), we get the estimate (i).

Similarly, applying Abel's transformation, the first part of estimate (ii) can be proved with the help of (3.2) and Lemma 2(ii). The second part of this estimate can be simply obtained by taking $t = 1/n$ in the first part.

§5. *Proof of Theorem* 1 — Let $S_n(x)$ be the nth partial sum of (2.1) at $t = x$ and $t_n(x)$ be its $(\bar{N}, p_n)$-mean. Then we have

$$t_n(x) - S = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \{S_\nu(x) - S\}$$

$$= \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \left\{ \frac{2}{\pi} \int_0^\pi \frac{\phi(t)}{t} \sin \nu t \, dt + o(1) \right\}$$

$$= \frac{2}{\pi P_n} \int_0^\pi \frac{\phi(t)}{t} N_n(t) \, dt + o(1)$$

$$= \frac{2}{\pi P_n} \left[\int_0^\pi \frac{\phi_1(t)}{t} N_n(t) \, dt - \int_0^\pi \phi_1(t) \, N'_n(t) \, dt \right] + o(1)$$

$$= \frac{2}{\pi P_n} (I_1 - I_2) + o(1), \text{ say} \qquad \ldots(5.1)$$

where

$$N_n(t) = \sum_{\nu=0}^{n} p_\nu \sin \nu t \quad \text{and} \quad N'_n(t) = \sum_{\nu=0}^{n} \nu p_\nu \cos \nu t.$$

Now, let us write

$$I_1 = \int_0^\pi \frac{\phi_1(t)}{t} N_n(t) \, dt$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \frac{\phi_1(t)}{t} N_n(t) \, dt$$

$$= I_{1,1} + I_{1,2}, \text{ say.} \qquad \ldots(5.2)$$

Using partial integration, it can be easily established that if (3.2) holds, then

$$\int_0^t | \phi_1(u) | \, du = o\left(t \log \frac{1}{t} \right) \text{ as } t \to 0.$$

Now, applying Lemma 3(iii), we have

$$| I_{1,1} | \leqslant \int_0^{1/n} | \phi_1(t) | \left| \frac{N_n(t)}{t} \right| dt$$

(*equation continued on p.* 1233)

$$= O\left(\frac{nP_n}{\log n} \int_0^{1/n} | \phi_1(t) | \, dt \right)$$

$$= o(P_n) \quad \text{as} \quad n \to \infty. \qquad \ldots(5.3)$$

Similarly, using Lemma 3(iii), we have

$$| I_{1,2} | \leqslant \int_{1/n}^{\pi} \frac{| \phi_1(t) |}{t} \, | N_n(t) | \, dt$$

$$= O\left(\frac{P_n}{\log n} \int_{1/n}^{\pi} \frac{| \phi_1(t) |}{t} \, dt \right)$$

$$= o(P_n) \quad \text{as} \quad n \to \infty, \text{ by (3.1)}. \qquad \ldots(5.4)$$

Collecting (5.2) to (5.4), we have

$$I_1 = o(P_n) \quad \text{as} \quad n \to \infty. \qquad \ldots(5.5)$$

Further, we write

$$I_2 = \int_0^{\pi} \phi_1(t) \, N'_n(t) \, dt$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \phi_1(t) \, N'_n(t) \, dt$$

$$= I_{2,1} + I_{2,2}, \text{ say}. \qquad \ldots(5.6)$$

Now, applying Lemma 4(i), we have

$$| I_{2,1} | \leqslant \int_0^{1/n} | \phi_1(t) | \; | N'_n(t) | \, dt$$

$$= O\left(\frac{nP_n}{\log n} \int_0^{1/n} | \phi_1(t) | \, dt \right)$$

$$= o(P_n) \quad \text{as} \quad n \to \infty. \qquad \ldots(5.7)$$

Again, Lemma 4(i) gives that

$$| I_{2,2} | \leqslant \int_{1/n}^{\pi} | \phi_1(t) | \; | N'_n(t) | \, dt$$

$$= O\left(\frac{P_n}{\log n} \int_{1/n}^{\pi} \frac{| \phi_1(t) |}{t} \, dt \right)$$

$$= o(P_n) \quad \text{as} \quad n \to \infty, \text{ by (3.1)}. \qquad \ldots(5.8)$$

Collecting (5.6) to (5.8), we get

$$I_2 = o(P_n) \text{ as } n \to \infty. \qquad \ldots(5.9)$$

Finally, the collection of (5.1), (5.5) and (5.9) gives

$$t_n(x) - S = o(1) \text{ as } n \to \infty.$$

This completes the proof of Theorem 1.

§6. *Proof of Theorem* 2 — Let $\bar{S}_n(x)$ be the nth partial sum of (2.2) at $t = x$ and $\bar{t}_n(x)$ be its $(\bar{N}, p_n)$-mean. Then we have

$$\bar{t}_n(x) - \bar{f}\left(x, \frac{1}{n}\right) = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \left\{\bar{S}_\nu(x) - \bar{f}\left(x, \frac{1}{n}\right)\right\}$$

$$= \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \left\{\frac{2}{\pi} \int_0^\pi \psi(t) \frac{\sin^2 \frac{1}{2}\nu t}{\tan \frac{1}{2} t} dt - \frac{1}{\pi} \int_{1/n}^{\pi} \frac{\psi(t)}{\tan \frac{1}{2} t} dt\right\} + o(1)$$

$$= \frac{2}{\pi P_n} \left\{ \int_0^{1/n} \frac{2\psi(t)}{t} L_n(t)\, dt - \int_{1/n}^{\pi} \frac{\psi(t)}{t} M_n(t)\, dt\right\} + o(1)$$

$$= \frac{2}{\pi P_n} (J_1 - J_2) + o(1) \qquad \ldots(6.1)$$

where $L_n(t) = \sum_{\nu=0}^{n} p_\nu \sin^2 \frac{1}{2}\nu t$ and $M_n(t) = \sum_{\nu=0}^{n} p_\nu \cos \nu t$.

Now, we observe that

$$J_1 = - \int_0^{1/n} \frac{2\Psi'(t)}{\log (1/t)} L_n(t)\, dt$$

$$= - \left[\frac{2\Psi(t)\, L_n(t)}{\log (1/t)}\right]_0^{1/n} + \int_0^{1/n} \frac{2\Psi(t)}{(\log (1/t))^2} \frac{L_n(t)}{t} dt$$

$$+ \int_0^{1/n} \frac{2\Psi(t)}{\log (1/t)} L'_n(t)\, dt.$$

Using Lemmas 3(iv), 3(v) and 3(i), we have

$$J_1 = o(P_n) + o\left(\frac{n^2 P_n}{\log n} \int_0^{1/n} \frac{t}{\log (1/t)} dt\right) + o\left(\frac{n P_n}{\log n} \int_0^{1/n} dt\right), \text{ by (3.3)}$$

$$= o(P_n) \text{ as } n \to \infty. \qquad \ldots(6.2)$$

Next

$$J_2 = -\int_{1/n}^{\pi} \frac{\Psi'(t)}{\log(1/t)} M_n(t)\,dt$$

$$= -\left[\frac{\Psi(t)}{\log(1/t)} M_n(t)\right]_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Psi(t)}{(\log(1/t))^2} \frac{M_n(t)}{t}\,dt$$

$$+ \int_{1/n}^{\pi} \frac{\Psi(t)}{\log(1/t)} M_n'(t)\,dt.$$

Applying Lemmas 4(ii) and 3(i), we have

$$J_2 = o(P_n) + o((P_n/\log n)\int_{1/n}^{\pi} t^{-1}dt) + o((P_n/\log n)\int_{1/n}^{\pi} t^{-1}dt), \text{ by (3.3)}$$

$$= o(P_n) \text{ as } n \to \infty. \qquad \ldots(6.3)$$

Combining (6.1) to (6.3), we have

$$\bar{t}_n(x) - \bar{f}\left(x, \frac{1}{n}\right) = o(1) \text{ as } n \to \infty.$$

This completes the proof of Theorem 2.

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