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HYDROMAGNETIC STABILITY OF A VORTEX SHEET IN COMPRESSIBLE, PERFECTLY CONDUCTING FLUIDS

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Hydromagnetic stability of a vortex sheet in perfectly conducting, inviscid and compressible fluids is examined when the acoustic properties of the fluids on the two sides of the vortex sheet are same. It is shown that it is sufficient, for the stability analysis, to consider only two-dimensional disturbances that propagate along the flow direction. The critical velocity at which instability first sets in, or beyond which there is no instability, or that at which the growth rate of instability is maximum, and the maximum growth rate of instability are all determined for magnetic fields of different strengths and orientations. In contrast to the incompressible case, for a given magnetic field there are usually two ranges of velocity discontinuity at which there is stability, and, for some special inclinations and magnitudes of the magnetic fields, there are even three ranges of stability. For a given orientation of magnetic fields, as their strength increases, the range of instability shifts towards larger velocities and the maximum growth rate of instability, after an initial decline, increases, indicating that a magnetic field, in the presence of compressibility, may be even destabilizing.

When the magnetic fields on the two sides of the vortex sheet are parallel, they have maximum stabilizing influence when they are along the flow. When the magnetic fields are perpendicular, their stabilizing influence is maximum when they are equally inclined with the flow.

1. INTRODUCTION

The stability of an interface (vortex sheet) between two fluids in relative motion in the presence of magnetic fields is of geophysical interest because such an interface seems to represent well the boundary between the solar wind and the earth's magnetosphere (Fejer 1964 and Talwar 1964). As the study of hydromagnetic stability, including the effect of compressibility of fluids, is very complicated, the discussion of the stability problem for a vortex sheet in the case of compressible fluids has been restricted to some limiting cases (Fejer 1964, Sen 1964, Wang and Maslen 1964, Jindia and Chakraborty 1972).

Sen (1964) and Fejer (1964) have found that a small compressibility of fluids (the speed of sound much greater than Alfvén speed) has always a destabilizing influence on an otherwise marginally stable equilibrium state of motion for arbitrary magnetic fields and densities which may be discontinuous across the vortex sheet. Sen has also examined the stability of the vortex sheet when the acoustic properties of the fluids and the magnetic fields on its two sides are identical in the steady state and the fluids are highly compressible (that is, when the speed of sound is much smaller than Alfvén speed; Sen actually takes sound speed to be zero). Fejer has considered the effect of arbitrary compressibility of fluids on the stability of the vortex sheet in the cases when the magnetic fields on its two sides are: (a) same in magnitude and direction, (b) same in magnitude but perpendicular to each other. He has found that, for a given velocity discontinuity across the vortex sheet, a critical magnetic field (Alfvén velocity) exists below which there is instability. His results may be equivalently interpreted to mean the existence of a critical velocity discontinuity (Mach number) for a given magnetic field; the vortex sheet is unstable or stable according as the velocity discontinuity is more or less than this critical value. Thus he finds stability in a single velocity range.

When compressibility of fluids is very high, the case (a) in Fejer's paper becomes identical with the corresponding problem considered by Sen. In the special case when the magnetic field is along the flow direction and disturbances propagate along the flow, Sen has found that, for high compressibility of fluids, the vortex sheet is always stable for any velocity discontinuity and Alfvén speed. Fejer's calculations, for this special case and zero sound speed, show that the vortex sheet is unstable when the velocity discontinuity is greater than Alfvén speed. Gerwin (1968) points out that the origin of the discrepancy in the results of Fejer and Sen may be that either (or both) of these studies fail to check that the modes whose stability is being considered actually remain bounded at large distances from the vortex sheet.

Sen has found that the effect of high compressibility is to give rise to two domains of stability in contrast to a single domain in the incompressible case. Fejer's calculations for arbitrary compressibility have shown that there is a critical magnetic field for a given velocity of discontinuity and thus, as explained before, there is only a single domain of stability (velocity range in which the vortex sheet is stable).

In view of the above discrepancies between the results of Sen and Fejer, it seems to be of some interest to examine the hydromagnetic stability of a vortex sheet in compressible fluids, carefully checking if all the unstable modes being considered remain bounded at large distances from the vortex sheet. In the present paper we study the stability problem in cases (a) and (b) of Fejer's paper for arbitrary compressibility. Fejer has given results in case (a) for magnetic fields parallel to the flow direction. He has considered in case (b) two orientations of the magnetic fields but has given results only for the case when the magnetic field on one side of the vortex sheet is parallel to the fluid velocity. We consider the stability of the vortex sheet, with the discontinuity in velocity taking different values, in both the cases for a number of strengths and orientations of the magnetic fields relative to the flow direction. We show that it is sufficient for stability analysis to consider only two-dimensional disturbances propagating along the flow, since the results for disturbances propagating at an angle with the flow follow from such a consideration. Consequently we consider only two-dimensional disturbances.

In case (a), when the fluids are highly compressible and the magnetic fields are along the flow, our calculations show that the vortex sheet is always stable, thus confirming Sen's result in contrast to Fejer's finding. For arbitrary compressibility of fluids we find that there are, in most of the cases studied by us, two regions of stability and in a few cases, for special orientations and magnitudes of magnetic fields, there are even three domains of stability. In this respect also, our finding for arbitrary compressibility differs from that of Fejer, but is similar to the result obtained by Sen for highly compressible fluids. For magnetic fields of different strengths and orientations we determine the instability domains [ranges of velocity discontinuities (Mach numbers) for which the vortex sheet is unstable], velocities at which instability is maximum, and the corresponding growth rates of instability.

2. Formulation of the Problem and Dispersion Relation

The system of rectangular Cartesian axes is so chosen that in the steady state we have a plane vortex sheet at $z = 0$ in compressible, inviscid and perfectly conducting fluids. The uniform velocity, density, magnetic field and sound speed for $z < 0$ are given by $\mathbf{v} = (U, 0, 0)$, ρ, $\mathbf{H} = (H_x, H_y, 0)$ and a, respectively, while the corresponding quantities for $z > 0$ are $\mathbf{v} = (U_0, 0, 0)$, ρ_0, $\mathbf{H}_0 = (H_{x0}, H_{y0}, 0)$ and a_0.

The stability of the vortex sheet against small three-dimensional disturbances in which the perturbation q' in any physical quantity q is of the form

$$q' = \hat{q}(z)\, e^{i(\alpha x + \beta y - \alpha c t)}$$

is studied. The wavenumbers α and β are positive and c is the complex wave velocity. A complex c with positive imaginary part implies instability of the vortex sheet.

Using the mks system of units and solving the linearized perturbation equations subject to appropriate boundary condition, we obtain the dispersion relation:

$$\frac{m}{\rho\left[(U-c)^2 - \dfrac{\mu(\Upsilon\,.\,\mathbf{H})^2}{\alpha^2\rho}\right]} = -\frac{m_0}{\rho_0\left[(U_0-c)^2 - \dfrac{\mu(\Upsilon\,.\,\mathbf{H}_0)^2}{\alpha^2\rho_0}\right]} \qquad \ldots(1)$$

where

$$m^2 = \gamma^2 - \frac{\alpha^2}{a^2}(U-c)^2\left\{1 + \frac{V_A^2}{a^2} - \frac{\mu(\underline{\gamma}\,.\,\mathbf{H})^2}{\alpha^2\rho(U-c)^2}\right\}^{-1} \quad ...(2)$$

$$m_0^2 = \gamma^2 - \frac{\alpha^2}{a_0^2}(U_0-c)^2\left\{1 + \frac{V_{A_0}^2}{a_0^2} - \frac{\mu(\underline{\gamma}\,,\,\mathbf{H}_0)^2}{\alpha^2\rho_0(U_0-c)^2}\right\}^{-1} \quad ...(3)$$

$$V_A^2 = \frac{\mu H^2}{\rho},\ V_{A_0}^2 = \frac{\mu H_0^2}{\rho_0}$$

and $\underline{\gamma} = \alpha\mathbf{e}_x + \beta\mathbf{e}_y$ is the vector wavenumber of disturbances. m and m_0 occurring in eqn. (1) shall be taken with non-negative real parts to ensure that disturbances remain bounded as $|z| \to \infty$.

After some straightforward reduction, we may write the dispersion relation as

$$\rho^2[(U-c)^2\cos^2\phi - V_A^2\cos^2\psi]^2\left[1 - \frac{(U_0-c)^2\cos^2\phi}{a_0^2}\times\left\{1 + \frac{V_{A_0}^2}{a_0^2} - \frac{V_{A_0}^2\cos^2\psi_0}{(U_0-c)^2\cos^2\phi}\right\}^{-1}\right]$$

$$= \rho_0^2[(U_0-c)^2\cos^2\phi - V_{A_0}^2\cos^2\psi_0]^2 \times\left[1 - \frac{(U-c)^2\cos^2\phi}{a^2}\left\{1 + \frac{V_A^2}{a^2} - \frac{V_A^2\cos^2\psi}{(U-c)^2\cos^2\phi}\right\}^{-1}\right] \quad ...(4)$$

where ϕ is the angle that the direction of propagation of disturbances makes with the flow direction, and ψ and ψ_0 are the angles that the magnetic fields $\mathbf{H}$, $\mathbf{H}_0$ make with the direction of propagation for $z < 0$ and $z > 0$, respectively.

In the special case when the disturbances propagate along the direction of flow and the magnetic fields are also along the same direction ($\phi = \psi = \psi_0 = 0$), the dispersion relation [eqn. (1)] reduces to that in Wang and Maslen (1964). It may be remarked that the dispersion relation (4) also reduces to the one obtained by Fejer (1964) in the appropriate limit.

We now choose a frame of reference in which the fluid in the region $z > 0$ is at rest in equilibrium. The dispersion relation (4) then reduces to (in the dimensionless form):

$$[(M-c_1)^2\cos^2\phi + V^2(M-c_1)^2\cos^2\phi - V^2\cos^2\psi - (M-c_1)^4\cos^4\phi]$$
$$\times\ [c_1^2\cos^2\phi - V_0^2\cos^2\psi_0]^2\,[A^2c_1^2\cos^2\phi + V_0^2c_1^2\cos^2\phi - A^2V_0^2\cos^2\psi_0] =$$

(*equation continued on p.* 1240)

$$= l[A^2c_1^2 \cos^2\phi + V_0^2c_1^2\cos^2\phi - A^2V_0^2\cos^2\psi_0 - c_1^4\cos^4\phi] \times [(M-c_1)^2\cos^2\phi - V^2\cos^2\psi]^2 \times [(M-c_1)^2\cos^2\phi + V^2(M-c_1)^2\cos^2\phi - V^2\cos^2\psi] \quad ...(5)$$

where the dimensionless quantities $c_1 = c/a$, $M = U/a$, $V = V_A/a$, $V_0 = V_{A0}/a_0$, $l = (\rho/\rho_0)^2$ and $a_0/a = A$ have been used.

3. Discussion of the Dispersion Relation

We note that if we put $\bar{M} = M\cos\phi$ and $\bar{c}_1 = c_1\cos\phi$ in eqn. (5) and compare the resulting equation with it, we can easily see that the eigen value (c_1) for a three-dimensional disturbance for a given magnetic field and velocity discontinuity (M) is very simply related ($\bar{c}_1 = c_1\cos\phi$) to the eigen value ($\bar{c}_1$) for a two-dimensional disturbance propagating along the flow ($\phi = 0$) for the same magnetic field but a different velocity discontinuity ($\bar{M}$). Consequently any instability present for three-dimensional disturbances will be present for two-dimensional disturbances also. It is thus sufficient to consider the stability of the vortex sheet with respect to two-dimensional disturbances and we shall consider only such disturbances in further discussion.

In the analysis of the dispersion relation we consider, apart from two-dimensional disturbances ($\phi = 0$), the special case when the acoustic properties of the media and the magnetic field magnitudes on the two sides of the vortex sheet are the same. We, therefore, take $l = A = 1$ and $V_0 = V$. The dispersion relation then takes the form of an algebraic equation of ninth degree in c_1 which is given in Appendix [eqn. ($A1$)].

The roots of this ninth degree equation have been computed numerically for different values of the parameters M, V, ψ and ψ_0. An admissible complex root c_1 of this equation with positive imaginary part will imply instability of the vortex sheet. A complex root is admissible if it also satisfies eqn. (1) (in its dimensionless form and in the special case when $\rho = \rho_0$, $a = a_0$, $\phi = 0$, $H = H_0$, $U_0 = 0$) with m and m_0 having non-negative real parts. All the complex roots have been duly tested for admissibility and only the admissible roots have been considered in further analysis.

Case (a) : Parallel Magnetic Fields ($\psi = \psi_0$)

For a choice of V from the set of values 0.1, 0.5, 1.0, 2.0 and 5.0 and of ψ from the set of values 0°, 15°, 30°, 45°, 60°, 75° and 90°, the roots c_1 of eqn. ($A1$) have been determined numerically for different values of Mach number M. The Mach number M_1, at which instability first sets in, and M_2, beyond which there is no instability, have been found. We also determine Mach number M_3 for which the growth rate of instability is maximum and, therefore, c_1, an admissible complex root of ($A1$) has maximum positive imaginary part c^*_{1i}. The value of c^*_{1i}, which is a measure of the growth rate of maximum instability, has also been observed. In this way, as the numerical computations cover the range of values of V and ψ as given above, the variations of the important stability characteristics with magnetic field strength (V) and its orientation (ψ) are obtained.

The results of numerical computation are displayed in Figs. 1 – 5. No instability is noted for $\psi = 0°$ (magnetic fields parallel to the flow) and $V \geqslant 1$. The variations of critical Mach numbers (velocities) M_1 and M_2 with ψ, for different magnetic field strengths (V), are shown in Figs. 1 and 2, respectively. The variation of the critical Mach number M_3 at which instability is maximum is shown in Fig. 3, while that of the corresponding maximum growth rate of instability c^*_{1i} is shown in Fig. 4.

From Figs. 1 – 4 we conclude that, with the increase in the angle of inclination ψ of the magnetic field with the flow direction, but for the fixed magnetic field strength (V) there is a decrease in the critical Mach number M_1 but the critical Mach number M_2 increases steadily. The range of instability thus increases with the inclination (ψ) of a magnetic field, of given strength, with the flow direction. The Mach number M_3 at which instability is maximum does not vary appreciably with ψ for a given strength (V) of the magnetic field. The maximum growth rate of instability c^*_{1i}, however, shows an appreciable increase [particularly for large magnetic fields (V)] with ψ.

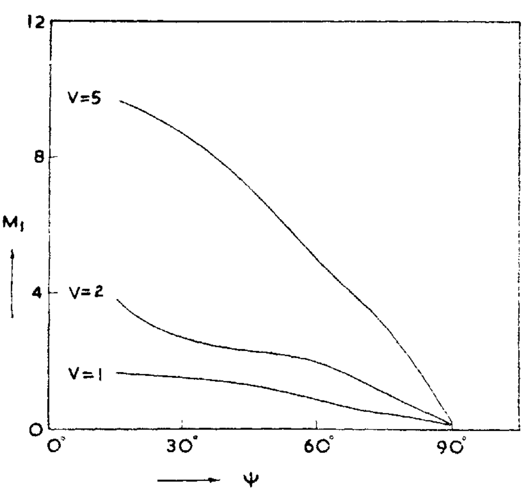

FIG. 1. Variation of the Mach number M_1 at which instability first sets in, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field with the flow direction in the case of parallel magnetic fields.

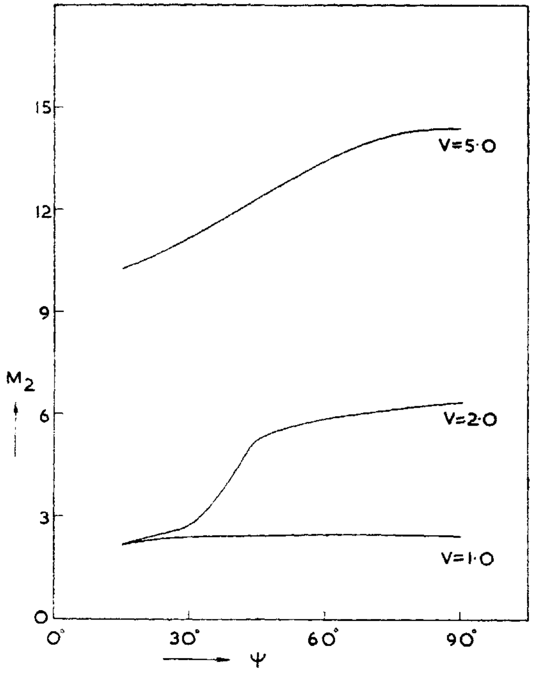

FIG. 2. Variation of the Mach number M_2 beyond which there is no instability, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field with the flow direction in the case of parallel magnetic fields.

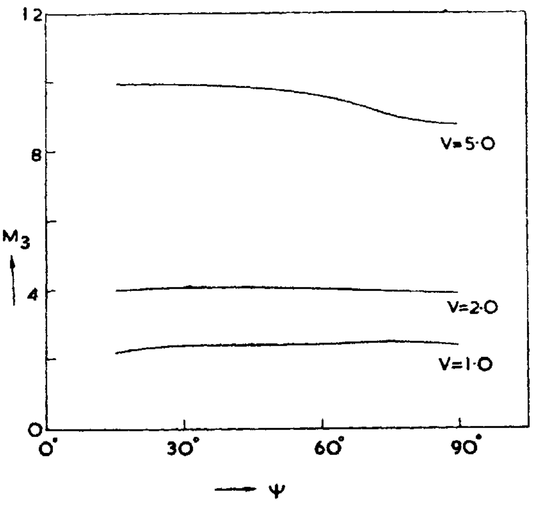

FIG. 3. Variation of the Mach number M_3 at which the growth rate of instability is maximum, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field with the flow direction in the case of parallel magnetic fields.

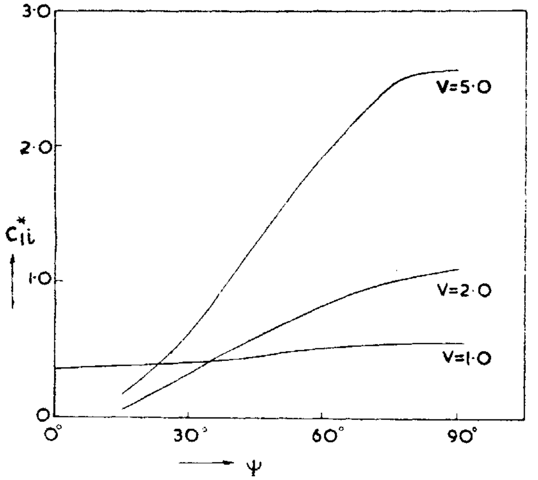

FIG. 4. Variation of the maximum growth rate of instability c^*_{1i}, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field with the flow direction in the case of parallel magnetic fields.

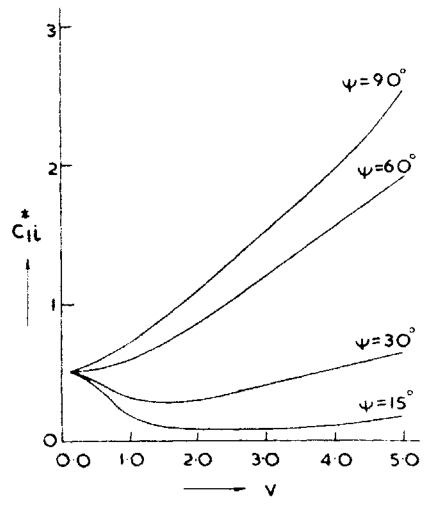

FIG. 5. Variation of the maximum growth rate of instability c^*_{1i}, for fixed inclination ψ of the magnetic field with the flow direction, with the magnetic field strengths (V) in the case of parallel magnetic fields.

For $V = 2.0$ and $\psi = 30°$ or $45°$, we find two ranges of instability $M_1 < M < M_4$ and $M_5 < M < M_2$. The vortex sheet is stable in the ranges $M < M_1$, $M_4 < M < M_5$ and $M > M_2$. For $\psi = 30°$, $M_4 = 2.90$, $M_5 = 3.45$, and for $\psi = 45°$, $M_4 = 2.50$, $M_5 = 2.81$.

From Figs. 1 – 3, it is also evident that, for a given angle of inclination (ψ), M_1, M_2 and M_3 increase as the magnitude of the magnetic field (V) increases. Thus the range of instability shifts towards larger velocities as the strength of magnetic field increases. Fig. 5 shows the variation of c^*_{1i} with V for fixed angles of inclination ψ. For some ψ, which are not large, the growth rate of maximum instability c^*_{1i} first shows a decline and then an increase with the increase of magnetic field (V). But, for comparatively large ψ, c^*_{1i} only increases with V. Thus, for compressible fluids, the magnetic field may have a destabilizing influence.

Case (b) : Perpendicular Magnetic Fields ($\psi - \psi_0 = 90°$)

Equation ($A1$) has been solved numerically for roots c_1 for $0 < \psi < 90°$. As in case (a), the variations of M_1, M_2 with ψ for different values of V are shown in Figs. 6 and 7, respectively, while those of M_3 and c^*_{1i} are shown in Figs. 8 and 9, respectively. All these figures are symmetric about $\psi = 45°$. For a given V, M_1 does not vary significantly with ψ while M_2 decreases in the range $0 < \psi < 45°$ and increases in the range $45° < \psi < 90°$. The variations of M_1, M_2 are such that the instability range decreases as ψ approaches 45° and becomes minimum at $\psi = 45°$. M_3 and c^*_{1i} remain nearly constant for small values of V but for higher values of V they decrease as ψ approaches 45°. The maximum growth rate of instability c^*_{1i} is also thus minimum at $\psi = 45°$. From the Figs. 6 – 8, it is again evident that, when the angle of inclination ψ is fixed, M_1, M_2 and M_3 increase with V (magnetic field). The

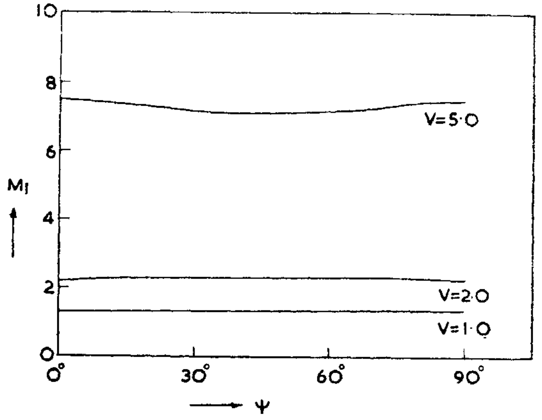

FIG. 6. Variation of the Mach number M_1 at which instability first sets in, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field (on one side of the vortex sheet) with the flow direction in the case of perpendicular magnetic fields.

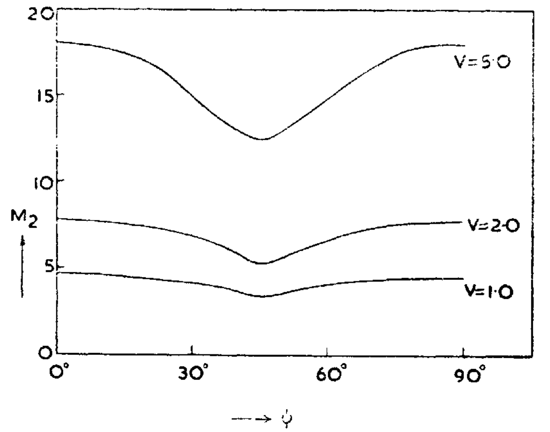

FIG. 7. Variation of the Mach number M_2 beyond which there is no instability, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field (on one side of the vortex sheet) with the flow direction in the case of perpendicular magnetic fields.

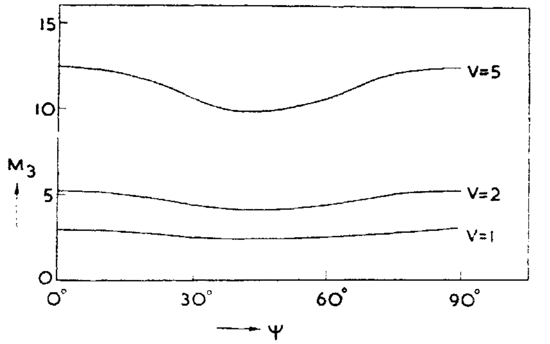

FIG. 8. Variation of Mach number M_3 at which the growth rate of instability is maximum, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field (on one side of the vortex sheet) with the flow direction in the case of perpendicular magnetic fields.

range of instability thus shifts toward larger velocities of discontinuity as the strength of magnetic field increases. Fig. 10 shows the variation of c_{1i}^* with V for fixed angles of inclination ψ, and it is clear from the figure that, as magnetic field (V) increases, for any fixed value of ψ, the maximum growth rate of instability c_{1i}^* first shows a small decline and then a considerable rise in value. Thus the magnetic field strength, for compressible fluids, may have a destabilizing effect also.

As in case (a) of parallel magnetic fields, we find that here too, for $V = 2.0$ and different values of ψ, there are two ranges of instability $M_1 < M < M_4$ and $M_5 < M < M_2$. The values of M_4 and M_5 are given in Table I.

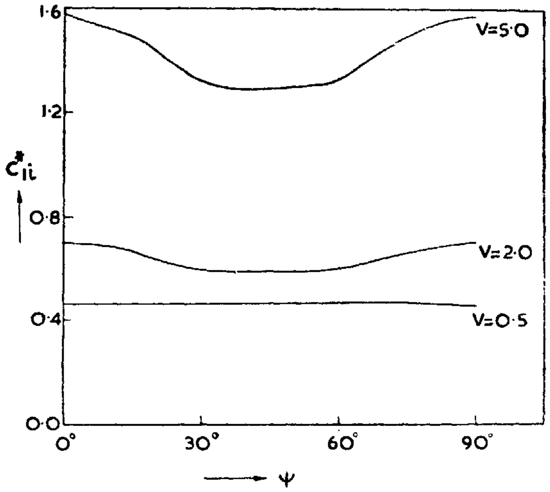

FIG. 9. Variation of the maximum growth rate of instability c^*_{1i}, for fixed magnetic field strengths (V), with the inclination ψ of the magnetic field (on one side of the vortex sheet) with the flow direction in the case of perpendicular magnetic fields.

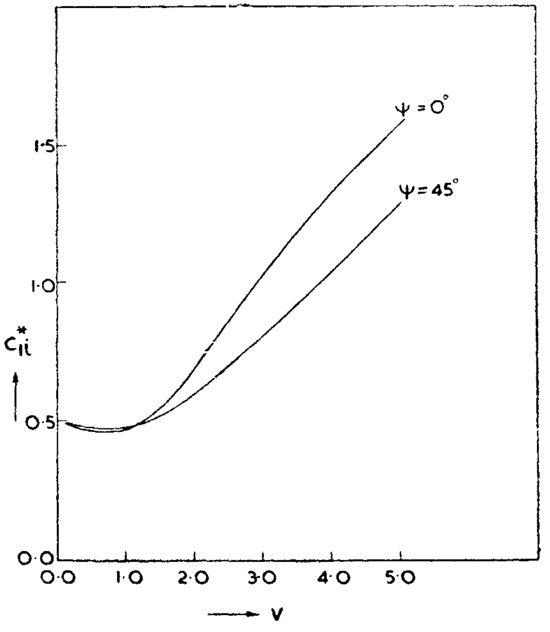

FIG. 10. Variation of the maximum growth rate of instability c^*_{1i}, for fixed inclination ψ of the magnetic field (on one side of the vortex sheet) with the flow direction, with the magnetic field strength (V) in the case of perpendicular magnetic fields.

TABLE I

ψ	0°	15°	30°	45°	60°	75°	90°
M_4	2.90	2.85	2.70	2.50	2.70	2.85	2.90
M_5	3.05	3.00	2.85	2.82	2.85	3.00	3.05

4. CONCLUSION

In case (a), when the magnetic fields are parallel to the direction of flow ($\psi = \psi_0 = 0$) and $V \geqslant 1$, the vortex sheet is always stable. When $V < 1$, the vortex sheet is unstable only for $M_1 < M < M_2$. Thus for all V, the vortex sheet is stable for sufficiently large M. As M may be taken as a measure of compressibility of the fluids, the vortex sheet, in the present case, is stable when the fluids have sufficiently large compressibility. This result is in conformity with Sen's finding but it contradicts Fejer's conclusion that the vortex sheet is unstable for a fluid of high compressibility when $M > V$.

For a given V, as ψ increases from 0° to 90°, both instability range and the maximum growth rate of instability c_{1i}^* increase in case (a). The parallel magnetic fields have thus the maximum stabilizing influence when they are along the direction of flow. Similarly, when the magnetic fields are orthogonal, they have maximum stabilizing influence when they are equally inclined to the direction of flow. In both the cases (a) and (b), for a given ψ, as V increases, the instability range shifts towards larger velocities. The magnetic field may have a destabilizing influence in the presence of compressibility for larger values of velocity discontinuity (M).

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APPENDIX

(Dispersion relation in the form of ninth degree algebraic equation in c_1).

$$
\begin{aligned}
&2M(1+V^2)c_1^9 + [V^2(2V^2+3)(\cos^2\psi - \cos^2\psi_0) - 9M^2(1+V^2)]c_1^8 \\
&+ 4M[4M^2(1+V^2) - (1+V^2)^2 - V^2(2V^2+3)(\cos^2\psi - \cos^2\psi_0)]c_1^7 \\
&+ [14M^2(1+V^2)^2 - 14M^4(1+V^2) - 2V^2(1+V^2)^2(\cos^2\psi - \cos^2\psi_0) \\
&+ 6M^2V^2(2V^2+3)(\cos^2\psi - \cos^2\psi_0) - V^4(V^2+3)(\cos^4\psi - \cos^4\psi_0)]c_1^6 \\
&+ 2M[V^2(1+V^2)(2V^2+3)(\cos^2\psi - \cos^2\psi_0) - 2M^2V^2(2V^2+3)(\cos^2\psi \\
&- \cos^2\psi_0) + V^2(1+V^2)(2V^2+3)\cos^2\psi + 3V^2(1+V^2)\cos^2\psi_0 \\
&+ V^4(V^2+3)(\cos^4\psi - \cos^4\psi_0) - V^4(V^2+3)\cos^4\psi_0 - 10M^2(1+V^2)^2 \\
&+ 3M^4(1+V^2)]c_1^5 + [M^4V^2(2V^2+3)(\cos^2\psi - \cos^2\psi_0) \\
&- M^2V^2(1+V^2)(2V^2+3)(\cos^2\psi - \cos^2\psi_0) - 15M^2V^2(1+V^2)\cos^2\psi_0 \\
&- 5M^2V^2(1+V^2)(2V^2+3)\cos^2\psi - M^2V^4(V^2+3)(\cos^4\psi - \cos^4\psi_0) \\
&+ 5M^2V^4(V^2+3)\cos^4\psi_0 + V^4(1+V^2)(V^2+3)(\cos^4\psi - \cos^4\psi_0) \\
&+ V^6(\cos^6\psi - \cos^6\psi_0) + 15M^4(1+V^2)^2 - M^6(1+V^2)]c_1^4 \\
&- 2M[2M^2V^4(V^2+3)\cos^4\psi_0 + V^4(1+V^2)(V^2+3)(\cos^4\psi - \cos^4\psi_0) \\
&+ 2V^4(2V^2+3)\cos^2\psi\cos^2\psi_0 - 10M^2V^2(1+V^2)\cos^2\psi_0 \\
&- 2M^2V^2(1+V^2)(2V^2+3)\cos^2\psi + 3M^4(1+V^2)^2 - 2V^6\cos^6\psi_0]c_1^3 \\
&+ [M^4V^4(V^2+3)\cos^4\psi_0 + M^2V^4(1+V^2)(V^2+3)(\cos^4\psi - \cos^4\psi_0) \\
&+ 6M^2V^4(2V^2+3)\cos^2\psi\cos^2\psi_0 - V^6(V^2+3)\cos^2\psi\cos^2\psi_0(\cos^2\psi \\
&- \cos^2\psi_0) - 6M^2V^6\cos^6\psi_0 - V^6(1+V^2)(\cos^6\psi - \cos^6\psi_0) \\
&- 15M^4V^2(1+V^2)\cos^2\psi_0 - M^4V^2(1+V^2)(2V^2+3)\cos^2\psi \\
&+ M^6(1+V^2)^2]c_1^2 + 2M[2M^2V^4\cos^4\psi_0 + V^4(1+V^2)(\cos^4\psi - \cos^4\psi_0) \\
&+ 2V^4\cos^4\psi - 2M^2V^2(2V^2+3)\cos^2\psi + 3M^4(1+V^2)]V^2\cos^2\psi_0\, c_1 \\
&- [M^4V^4\cos^4\psi_0 + 2M^2V^4\cos^4\psi + M^2V^4(1+V^2)(\cos^4\psi - \cos^4\psi_0) \\
&- V^6\cos^2\psi(\cos^4\psi - \cos^4\psi_0) - M^4V^2(2V^2+3)\cos^2\psi \\
&+ M^6(1+V^2)]V^2\cos^2\psi_0 = 0. \qquad \ldots(\mathrm{A1})
\end{aligned}
$$