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THE LARGEST r FOR WHICH $(n+k)!/n!(k+r)!$ IS AN INTEGER

HANSRAJ GUPTA, F.N.A.
402 *Mumfordganj, Allahabad* 211002
AND
KULDIP SINGH
University of New Brunswick, Fredericton, N.B. E3B5A3, Canada

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For two given positive integers n and k, $1 \leqslant k \leqslant n$, let $r = r(n, k)$ denote the largest nonnegative integer for which $(k+r)!$ divides $(n+k)!/n!$ and $R(n)$ the largest $r(n, k)$ in the range $1 \leqslant k \leqslant n$. In this paper, we study some properties of $r(n, k)$ and $R(n)$. In particular, we show that $r(3^s, k) \leqslant 2$ for each positive integer s and $k \leqslant 3^s$. Moreover, for each k, we can find an n such that $r(n, k) \geqslant m$, where m is any pre-assigned positive integer. This shows that $R(n)$ does not tend to infinity with n, solving a problem of Professor Erdös.

1. INTRODUCTION

In what follows, small letters denote positive integers and p's are primes, unless stated otherwise. We write $(u; v)$ for $\binom{u}{v}$; and $\text{pot}_p\, n$ (read potency of n to the base p) to denote the largest power of p dividing n.

Given n, k $(1 \leqslant k \leqslant n)$, we denote by $r(n, k)$ the largest r for which $(n+k)!/n!(k+r)!$ is an integer. Clearly $r \geqslant 0$. We also let $R(n)$ denote the max $r(n, k)$, $1 \leqslant k \leqslant n$.

The object of this paper is to present some properties of $r(n, k)$. Incidently our results also answer a problem suggested by Professor P. Erdös to the second author on May 15, 1979.

Evidently, the expression in the title can be written in the form :

$$(n+k; k)/(k+1)(k+2)\ldots(k+r)$$

where the denominator is taken as 1 when $r = 0$; and also in the form :

$$(n+1)(n+2)\ldots(n+k)/(k+r)!$$

2. SOME GENERAL RESULTS

Theorem 1 — For any $c \geqslant 1$ and $p \geqslant 2$,

$$r(p^c, k) = 0$$

whenever k is of the form $sp - 1$, and $s \leqslant p^{c-1}$.

PROOF: For any j, $1 \leqslant j \leqslant k < p^c$,

$$\text{pot}_p (p^c + j) = \text{pot}_p (j).$$

Therefore

$$\text{pot}_p (p^c + k; k) = \sum_{j=1}^{k} \text{pot}_p (p^c + j) - \sum_{i=1}^{k} \text{pot}_p (j) = 0.$$

This implies that

$$p \nmid (p^c + k; k) \text{ for any } k < p^c.$$

For each k of the form $sp - 1$, $p \mid (k + 1)$. For any such k with $s \leqslant p^{c-1}$ therefore

$$(k + 1) \nmid (p^c + k; k).$$

Hence the theorem follows.

Corollary — For each k of the form $sp - 1$, $1 \leqslant s \leqslant p^{c-1}$ and $(m, p) = 1$

$$r(mp^c, k) = 0.$$

Theorem 2 — For any $p \geqslant 2$ and $k \not\equiv -1 \pmod{p}$; and $c \geqslant 1$,

$$r(p^c, k) > 0.$$

PROOF: We have

$$(k + 1)(p^c + k + 1; k + 1) = (p^c + k + 1)(p^c + k; k).$$

Since $(p^c + k + 1, k + 1) = (p^c, k + 1) = 1$, it follows that

$$(k + 1) \mid (p^c + k; k);$$

and the theorem follows.

Theorem 3 — For any odd prime p and $c \geqslant 1$,

$R(p^c)$ cannot exceed $p - 1$.

PROOF: For $1 \leqslant k \leqslant p^c$,

$$\text{pot}_p (p^c + k; k) = 0;$$

while p divides $(k + 1)(k + 2) \dots (k + p)$. Hence

$(k + 1)(k + 2) \dots (k + p)$ cannot divide $(p^c + k; k)$.

This proves the theorem.

3. The Special Case of $n = 2^c$, $c \geqslant 1$

At the very outset, we might observe that

$$\text{pot}_2 (2^c + k; k) = 0 \text{ or } 1$$

according as $1 \leqslant k < 2^c$ or $k = 2^c$.

Hence (2^c+4) cannot possibly divide $(2^c+k;k)$ for any $c>1$.

$$(\text{pot}_2\,(2^c+4) \geqslant 2 \text{ for } c>1).$$

This means that $R(2^c)$ cannot exceed 3 for any $c>1$.

Since for $c=1$ or 2, it is easy to show that $R(2^c)=1$, it follows that $R(2^c) \leqslant 3$ for each $c \geqslant 1$.

In the following, we can therefore take $c>2$.

Theorem 4 — For $c>2$, and $1 \leqslant k \leqslant 2^c$,

$$r(2^c,k) = \begin{cases} 0 \text{ if } k \text{ is odd,} \\ 1 \text{ if } k \text{ is even but } < 2^c, \\ 1,2, \text{ or } 3 \text{ for } k=2^c. \end{cases}$$

PROOF: For k odd, the result follows from Theorem 1. For k even but $<2^c$,

$(2^c+k;k)$ is odd and $(k+1)$ divides it by Theorem 2, but $k+2$ being even cannot possibly divide it. Hence

$$r(2^c,k)=1 \text{ for } k \text{ even but } <2^c.$$

For $k=2^c$,

$$\text{pot}_2\,(2^c+k;k) = \text{pot}_2\,(2^{c+1};2^c)=1;$$

and we have

$$(2^c+1)\,(2^{c+1}+1;2^c+1) = (2^{c+1}+1)\,(2^{c+1};2^c).$$

Since $(2^c+1, 2^{c+1}+1)=1$;

$$(2^c+1) \mid (2^{c+1};2^c).$$

This means that

$$r(2^c,2^c) \text{ is at least } 1.$$

Now

$$(2^c+1)\,(2^c+2)\,(2^{c+1}+2;2^c+2) = (2^{c+1}+1)\,(2^{c+1}+2)\,(2^{c+1};2^c)$$

which is the same thing as

$$(2^{c-1}+1)\,(2^{c+1}+2;2^c+2) = (2^{c+1}+1)\,(2^{c+1};2^c).$$

Since

$$(2^{c-1}+1, 2^{c+1}+1) = (2^{c-1}+1, 3) = 1 \text{ or } 3;$$

and 2 is a primitive root of 3 and all its powers;

it follows that $(2^{c-1} + 1, 3) = 3$, if and only if

$$2^{c-1} \equiv -1 \pmod 3, \text{ i.e. } 2^c \equiv 1 \pmod 3;$$

i.e. if and only if

$$c \equiv 0 \pmod 2.$$

Hence for c odd,

$$(2^{c-1} + 1) \mid (2^{c+1}; 2^c).$$

Also $$2 \mid (2^{c+1}; 2^c)$$

and $$(2^{c-1} + 1, 2) = 1.$$

Hence $(2^{c+1}; 2^c)$ is divisible by $2(2^{c-1} + 1)$ i.e. $2^c + 2$.

Moreover, it has already been shown that

$$(2^c + 1) \mid (2^{c+1}; 2^c).$$

Since $(2^c + 1, 2^c + 2) = 1$, it follows that

$$(2^c + 1)(2^c + 2) \mid (2^{c+1}; 2^c).$$

This means that for c odd,

$$r(2^c, 2^c) \text{ is at least } 2.$$

For c even,

$$r(2^c, 2^c) = 1 \text{ if } \operatorname{pot}_3 (2^{c-1} + 1) > \operatorname{pot}_3 (2^{c+1}; 2^c)$$
$$\geqslant 2 \text{ if } \operatorname{pot}_3 (2^{c-1} + 1) \leqslant \operatorname{pot}_3 (2^{c+1}; 2^c).$$

Assume that c is either odd or if even then

$$\operatorname{pot}_3 (2^{c-1} + 1) \leqslant \operatorname{pot}_3 (2^{c+1}; 2^c).$$

Since

$$(2^c + 3, 2^c + 1) = (2^c + 3, 2) = 1;$$

and $$(2^c + 2, 2^c + 3) = 1,$$

to find the conditions under which

$$r(2^c, 2^c) = 3$$

it will be enough to consider the circumstances under which

$$(2^c + 3) \text{ divides } (2^{c+1}; 2^c).$$

Consider the identical relation

$$(2^c + 1)(2^c + 2)(2^c + 3)(2^{c+1} + 3; 2^c + 3)$$
$$= (2^{c+1} + 1)(2^{c+1} + 2)(2^{c+1} + 3)(2^{c+1}; 2^c).$$

Here $(2^c + 3, 2^{c+1} + 2) = 1 = (2^c + 3, 2^{c+1} + 3)$,

while $(2^c + 3, 2^{c+1} + 1) = (2^c + 3, 5)$.

This is 5 if and only if

$$2^c \equiv -3 \equiv 2 \pmod 5.$$

Since 2 is a primitive root of 5 and all its powers, for this congruence to hold, we must have

$$c - 1 \equiv 0 \pmod 4.$$

Unless c is of the form $1 + 4t$, therefore, $(2^c + 3, 5) = 1$, and $2^c + 3$ divides $(2^{c+1}; 2^c)$.

If c is of the form $1 + 4t$,

$2^c + 3$ will divide $(2^{c+1}; 2^c)$ if and only if

$$\text{pot}_5 (2^c + 3) \leqslant \text{pot}_5 (2^{c+1}; 2^c).$$

Notice further that for $c \equiv 1 \pmod 4$ c is odd and

$(2^c + 1)(2^c + 2)$ already divides $(2^{c+1}; 2^c)$.

It will thus be clear that with $c > 1$,

(i) $r(2^c, 2^c) = 1$, if c is even and $\text{pot}_3 (2^{c-1} + 1) > \text{pot}_3 (2^{c+1}; 2^c)$.

(ii) $r(2^c, 2^c) = 2$, if $c \equiv 1 \pmod 4$ and

$\text{pot}_5 (2^c + 3) > \text{pot}_5 (2^{c+1}; 2^c)$.

(iii) $r(2^c, 2^c) = 3$ otherwise.

4. The Case When n is a Power of 3

Recall that $R(3^c) \leqslant 2$, for each $c \geqslant 1$.

Theorem 5 — For $1 \leqslant k \leqslant 3^c$, $c \geqslant 1$;

$$r(3^c, k) = \begin{cases} 0 & \text{if } k \equiv 2 \pmod 3, \\ 1 & \text{if } k \equiv 1 \pmod 3, \\ 1 \text{ or } 2 & \text{if } k \equiv 0 \pmod 3. \end{cases}$$

Proof: If $k \equiv 2 \pmod 3$, $r(3^c, k) = 0$ by Theorem 1.

If $k \equiv 1 \pmod 3$,

in the identical relation

$$(k+1)(3^c+k+1; k+1) = (3^c+k+1)(3^c+k; k)$$

$(3^c+k+1, k+1) = 1$. Therefore

$(k+1)$ divides $(3^c+k; k)$, but $(k+2)$ does not because $(k+2)$ is a multiple of 3 while $(3^c+k; k)$ is prime to 3 as in the proof of Theorem 1 with $p = 3$. Hence

$$r(3^c, k) = 1 \text{ if } k \equiv 1 \pmod 3.$$

When k is of the form $3t$, $1 \leqslant t \leqslant 3^{c-1}$, it is easy to show that $(k+1)$ still divides $(3^c+k; k)$. We proceed to consider the conditions under which $(k+2)$ will also divide $(3^c+k; k)$. Using the usual technique, we find that in the identity,

$$(k+1)(k+2)(3^c+k+2; k+2)$$
$$= (3^c+k+1)(3^c+k+2)(3^c+k; k$$

we have

$$(3^c+k+2, k+2) = (3^c, k+2) = 1$$

but $$(3^c+k+1, k+2) = (3^c-1, k+2) = d \text{ (say)}.$$

If $d = 1$, then $(k+2)$ divides $(3^c+k; k)$ without doubt and so does $(k+1)$.

Hence $$r(3^c, 3t) = 2.$$

If however, $d > 1$, then $r(3^c, 3t) = 2$ if and only if $d \mid (3^c+k; k)$. Otherwise $r(3^c, 3t) = 1$.

We leave it to the reader to verify that for $n = 81$,

$$r(81, 3t) = 2$$

only in the following cases:

$$t = 3, 5, 7, 8, 9, 13, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25 \text{ and } 27.$$

5. The Problem of Erdös

Erdös asked if $R(n)$ tended to infinity with n. Our results show that it does not. For given any n however large, one can find a c such that $3^c > n$. But Theorem 5 shows that $R(3^c) \leqslant 2$ for all c.

We however note that given any m, there exists an n given by $(m+k)! - k$ such that $R(n) \geqslant r(n, k) \geqslant m$. Thus $R(n)$ can be made arbitrarily large.

We further remark that if there exists an absolute constant C such that $R(n) < C \log n$ for all n, then C must be > 1. We give a few examples of values of n for which $R(n) > \log n$:

$$R(6) = 3 > \log 6 = 1\cdot 79176 \ldots$$

$$R(19) = 4 > \log 19 = 2\cdot 94444 \ldots$$

$$R(42) = 5 > \log 42 = 3\cdot 73767 \ldots$$

$$R(127) = 6 > \log 127 = 4\cdot 84420 \ldots .$$

6. Computation of $r(n, k)$

Let

$$2 = p_1 < p_2 < \ldots < p_i < \ldots$$

be the set of primes.

Then, if $\text{pot}_{p_j} n = a_j \geqslant 0,$

we shall say that

$$V(n) = (a_1, a_2, \ldots, a_j, \ldots)$$

is the potency-vector of n.

Evidently, the potency-vector will have only a finite number of non-zero components.

We write

$$V(n) \geqslant V(m)$$

if every component of $V(n)$ is $\geqslant$ the corresponding element of $V(m)$. Clearly $V(n) \geqslant V(m)$ if and only if n/m is an integer.

If at least one component of $V(n)$ is less than the corresponding component of $V(m)$, then n/m is not an integer.

Some years back, S. P. Khare computed a table giving the potency-vectors of $n!$ for $2 \leqslant n \leqslant 1000$. Thus, it gave

$$V(35!) = (32, 15, 8, 5, 3, 2, 2, 1, 1, 1, 1, 0, 0, \ldots).$$

We have made use of this table to find our $r(n, k)$.

Thus for $n = 23$, $k = 12$; we needed besides the potency-vector of $35!$, the potency-vectors of $23!$ and $12!$, $13!$, $14!$ ... .

These are

$$V(23!) = (19, 9, 4, 3, 2, 1, 1, 1, 1, 0, 0, \ldots)$$

$$V(12!) = (10, 5, 2, 1, 1, 0, 0, 0, 0, 0, 0, \ldots)$$

$$V(13!) = (10, 5, 2, 1, 1, 1, 0, 0, 0, 0, 0, \ldots)$$

$$V(14!) = (11, 5, 2, 2, 1, 1, 0, 0, 0, 0, 0, \ldots)$$

$$V(15!) = (11, 6, 3, 2, 1, 1, 0, 0, 0, 0, 0, \ldots)$$

$$V(16!) = (15, 6, 3, 2, 1, 1, 0, 0, 0, 0, 0, \ldots).$$

We observe that

$$V(35!) > V(23!\,15!) = V(23!) + V(15!)$$
$$= (30, 15, 7, 5, 3, 2, 1, 1, 1, 0, 0, 0, \ldots)$$

but

$V(35!)$ is not greater than $V(23!\,16!)$ which is

$$= (34, 15, 7, 5, 3, 2, 1, 1, 1, 0, 0, 0, \ldots).$$

Hence

$$r(23, 12) = 15 - 12 = 3.$$

If the potency-tables were not available, we could have adopted the following course:

Take $n = 23$ as before.

For any given $k \leqslant 23$, we have to find here the largest r for which

$$24.25.26. \ldots (23 + k)/(k + r)!$$

is an integer. For $k = 1$:

We have $24 = 2.2.2.3$.

$24/1!$, $24/2!$, $24/3!$, $24/4!$ are integers but $24/5!$ is not.

Hence $r(23, 1) = 4 - 1 = 3$.

Since $24/4! = 1$, we get

$$24.25/4! = 5.5.$$

Evidently $24.25/5! = 5$ is an integer but $24.\ 25/6!$ is not.

Hence for $k = 2$, we have

$$r(23, 2) = 5 - 2 = 3.$$

At the next step, we have

$$24.25.26/5! = 5.\ 26 = 5.2.13.$$

Since $5.2.13/6$ is not an integer, $24.25.26/6!$ is not an integer.

Hence $r(23, 3) = 5 - 3 = 2$.

We proceed in the same manner till we are through with $k = 23$. The process is best presented in the following form:

k	$n+k$	Prime decomposition of $(n+k)$	$k+r$	r
1	24	$\check{2}.\check{2}.\check{2}.\check{3}$	1, 2, 3, 4	3
2	25	$\check{5}.\check{5}$	5	3
3	26	$\check{2}.\check{13}$		2
4	27	$\check{3}.\check{3}.\check{3}$	6	2
5	28	$\check{2}.\check{2}.\check{7}$	7	2
6	29	29		1
7	30	$\check{2}.\check{3}.\check{5}$	8, 9	2
8	31	31		1
9	32	$\check{2}.\check{2}.\check{2}.\check{2}.2$	10	1
10	33	$\check{3}.\check{11}$	11, 12, 13	3
11	34	2.17		2
12	35	$5.\check{7}$	14, 15	3

and so on.

At each step, we tick off if possible factors which go to make the next factorial and the next if possible and so on.

In the above, we have finally got for $k = 12$,

$$24.25.26. \ldots 35/15! = 29.31.2.2.17.5$$

and $$r(23, 12) = 15 - 12 = 3.$$

7. A Relation Between $r(p, k)$ and $r(p-1, k+1)$

The following example with $p = 29$, will make the relationship clear.

Proceeding as in the preceding section, we have the following table (see p. 1258).

Let k_0 denote the least k for which

$$(p+1)(p+2) \ldots (p+k)/(p-1)!$$

is an integer. Then the reader will readily see that:

For primes $p \geqslant 5$,

$$r(p, j) = r(p-1, j+1) + 1, \text{ for each } j \leqslant k_0 - 1,\ j > 0$$

and $$r(p, j) = p - 1 - j, \text{ for } k_0 \leqslant j \leqslant p - 1.$$

Also $$r(p-1, 1) = 0.$$

$n = 28$					$n = 29$			
k	$n+k$	Prime factors	$r+k$	r	k	$n+k$	$r+k$	r
1	29	29	1	0	1	30	1, 2, 3	2
2	30	2.3.5	2, 3	1	2	31		1
3	31	31		0	3	32	4, 5	2
4	32	2.2.2.2.2	4, 5	1	4	33	6	2
5	33	3.11	6	1	5	34		1
6	34	2.17		0	6	35	7, 8	2
7	35	5.7	7, 8	1	7	36	9, 10, 11	4
8	36	2.2.3.3	9, 10, 11	3	8	37		3
9	37	37		2	9	38		2
10	38	2.19		1	10	39	12, 13	3
11	39	3.13	12, 13	2	11	40		2
12	40	2.2.2.5		1	12	41		1
13	41	41		0	13	42	14, 15	2
14	42	2.3.7	14, 15	1	14	43		1
15	43	43		0	15	44	16, 17	2
16	44	2.2.11	16, 17	1	16	45	18, 19	3
17	45	3.3.5	18, 19	2	17	46		2
18	46	2.23		1	18	47		1
19	47	47		0	19	48	20	1
20	48	2.2.2.2.3	20	0	20	49	21, 22, 23	3
21	49	7.7	21, 22, 23	2	21	50		2
22	50	2.5.5		1	22	51	24, 25	3
23	51	3.17	24, 25	2	23	52	26	3
24	52	2.2.13	26	2	24	53		2
25	53	53		1	25	54	27, 28	3
26	54	2.3.3.3	27, 28, 29	3	26	55		2
27	55	5.11		2	27	56		1
28	56	2.2.2.7		1	28	57		0
					29	58	29, 30, 31	2

TABLES OF VALUES OF $r(n, k)$

$1 \leqslant k \leqslant n \leqslant 50$

$k \backslash n$	1	2	3	4	5	6	7	8	9	10
1	1	0	1	0	2	0	1	0	1	0
2		1	0	1	1	0	2	1	0	1
3			2	0	1	1	3	0	2	0
4				1	0	3	2	1	1	0
5					2	2	1	0	0	0
6						1	0	1	1	2
7							1	0	1	1
8								3	0	1
9									2	0
10										1

$k \backslash n$	11	12	13	14	15	16	17	18	19	20
1	2	0	1	0	1	0	2	0	1	0
2	1	0	1	3	0	1	1	0	1	1
3	1	0	2	2	1	0	2	0	2	0
4	1	1	1	2	0	1	1	1	1	0
5	3	0	2	1	1	0	2	0	2	2
6	2	1	1	0	2	1	1	1	1	1
7	2	0	1	2	1	0	2	0	1	0
8	1	0	2	3	0	1	1	0	1	1
9	1	1	2	2	0	0	2	0	2	0
10	0	1	1	2	1	1	1	1	1	1
11	3	0	1	1	0	0	2	0	2	0
12		2	0	1	1	1	1	1	1	3
13			1	0	2	0	2	0	3	2
14				1	1	1	1	2	2	1
15					2	0	1	1	2	0
16						3	0	1	1	2
17							2	0	1	1
18								3	0	1
19									4	0
20										3

$k \backslash n$	21	22	23	24	25	26	27	28	29	30
1	1	0	3	0	1	0	1	0	2	0
2	0	2	3	0	1	1	0	1	1	0
3	1	2	2	0	1	0	2	0	2	1
4	1	1	2	1	0	1	1	1	2	0
5	0	1	2	0	2	0	0	1	1	0
6	0	1	1	1	1	3	2	0	2	2
7	2	0	2	0	2	2	1	1	4	1
8	1	1	1	2	1	1	0	3	3	0
9	2	0	1	2	0	2	2	2	2	0
10	1	0	3	3	3	2	1	1	3	1
11	2	2	2	2	3	1	0	2	2	0
12	1	1	3	3	2	0	2	1	1	2
13	0	2	4	2	1	1	1	0	2	1
14	1	3	3	1	0	3	0	1	1	0
15	3	2	4	0	4	2	2	0	2	2
16	2	3	3	3	3	3	1	1	3	1
17	2	2	3	2	2	2	0	2	2	0
18	1	2	2	1	1	1	2	1	1	1
19	1	1	2	0	0	2	1	0	1	0
20	0	1	1	2	2	3	0	0	3	0
21	2	0	1	1	2	2	2	2	2	2
22		2	0	1	1	2	1	1	3	1
23			3	0	1	1	0	2	3	0
24				2	0	1	1	2	2	0
25					1	0	1	1	3	1
26						3	0	3	2	0
27							2	2	1	1
28								1	0	1
29									2	0
30										1

$k \diagdown n$	31	32	33	34	35	36	37	38	39	40
1	1	0	1	0	2	0	1	0	1	0
2	2	1	0	1	1	0	1	3	0	1
3	1	0	2	0	1	0	2	2	2	0
4	1	1	1	1	0	1	1	3	1	0
5	3	0	2	0	1	0	2	2	0	0
6	2	1	1	1	0	1	1	1	1	1
7	1	0	1	0	2	0	1	0	1	0
8	1	1	0	1	1	0	1	1	0	1
9	2	0	1	0	0	0	2	0	2	0
10	1	1	0	1	1	1	1	3	1	1
11	3	0	0	0	2	0	2	3	0	0
12	2	1	1	1	1	1	2	2	0	2
13	1	0	2	0	0	1	1	2	1	1
14	3	1	1	1	0	0	1	1	0	1
15	2	0	2	0	1	0	2	0	2	0
16	1	1	1	0	1	1	1	1	1	1
17	2	0	0	0	2	0	2	0	1	0
18	1	1	1	1	1	1	1	0	3	1
19	1	0	1	0	2	0	4	2	4	0
20	3	1	0	1	1	3	3	3	3	1
21	2	0	0	0	2	2	2	2	2	0
22	1	1	1	1	1	1	1	1	1	1
23	1	0	2	0	1	0	2	0	2	0
24	2	1	1	1	0	1	1	1	1	0
25	1	0	1	0	2	0	1	0	0	1
26	2	1	0	1	1	0	1	3	0	3
27	2	0	2	0	2	0	2	2	2	2
28	1	1	1	1	1	1	1	3	1	1
29	1	0	0	0	0	0	2	2	0	0
30	0	1	1	2	2	1	1	2	2	1
31	1	0	1	1	2	0	1	2	1	0
32		2	0	1	1	0	2	3	0	2
33			2	0	1	1	2	2	2	1
34				1	0	1	1	2	1	0
35					3	0	1	1	0	0
36						2	0	1	1	2
37							1	0	1	1
38								3	0	1
39									2	0
40										1

k \ n	41	42	43	44	45	46	47	48	49	50
1	2	0	1	0	1	0	3	0	1	0
2	1	0	1	1	0	2	2	0	1	1
3	1	0	2	0	1	1	2	0	2	0
4	1	1	1	2	0	1	3	1	1	0
5	2	0	1	2	0	2	3	0	1	0
6	1	0	2	1	1	2	2	1	0	1
7	2	1	1	0	1	1	3	0	2	0
8	1	0	3	2	0	2	3	1	1	1
9	2	2	2	1	1	2	5	0	2	0
10	1	1	1	0	1	4	5	1	1	1
11	3	0	4	0	3	4	4	0	2	0
12	2	3	3	3	3	3	3	1	1	1
13	2	2	2	2	2	2	2	0	0	0
14	1	1	1	1	1	1	1	1	1	0
15	2	0	2	0	2	0	2	0	4	4
16	1	1	1	1	1	1	1	3	3	3
17	2	0	2	0	0	0	4	2	3	2
18	1	1	1	1	1	3	3	3	2	1
19	2	0	2	0	2	2	3	2	1	0
20	1	1	1	1	1	2	2	2	0	3
21	2	0	3	0	3	1	1	2	2	2
22	1	2	4	4	2	0	2	1	1	2
23	1	3	3	3	1	1	3	0	3	1
24	2	2	2	2	0	2	2	2	2	0
25	4	1	1	4	1	1	4	1	1	2
26	3	0	3	3	0	3	3	0	1	1
27	2	2	2	2	2	2	2	2	0	2
28	1	1	1	3	1	1	3	1	1	1
29	2	0	4	2	0	2	3	0	2	0
30	1	3	3	1	1	2	2	1	1	2
31	3	2	3	0	1	1	4	0	1	2
32	2	2	3	0	0	3	3	0	2	2
33	1	2	2	2	2	2	2	2	1	1
34	1	1	1	1	1	1	5	1	0	0
35	3	0	3	0	0	4	4	0	4	2

(continued)

$k \backslash n$	41	42	43	44	45	46	47	48	49	50
36	2	2	2	2	3	3	3	3	3	3
37	2	1	1	2	4	2	5	2	2	2
38	1	0	2	3	3	4	4	1	1	1
39	1	1	3	2	3	3	4	0	2	0
40	0	3	2	2	2	3	3	3	1	1
41	4	2	1	1	2	2	4	2	0	2
42		5	0	1	1	3	3	1	1	2
43			4	0	4	2	2	0	3	1
44				3	3	1	1	2	2	0
45					2	0	1	1	2	2
46						3	0	1	1	2
47							2	0	1	1
48								3	0	1
49									4	0
50										4

$r(n, k)$ *for* $n = 119, 127$

k	$r(119, k)$	$r(127, k)$	k	$r(119, k)$	$r(127, k)$
1	4	1	16	0	1
2	3	2	17	3	2
3	2	2	18	2	2
4	2	1	19	1	1
5	1	1	20	0	4
6	0	2	21	4	3
7	2	1	22	3	2
8	1	5	23	2	3
9	2	4	24	2	2
10	2	3	25	2	1
11	2	2	26	1	3
12	1	1	27	0	2
13	0	4	28	3	1
14	2	3	29	3	3
15	1	2	30	2	2

(continued)

k	$r(119, k)$	$r(127, k)$	k	$r(119, k)$	$r(127, k)$
31	2	1	66	1	3
32	1	2	67	0	2
33	0	2	68	1	1
34	4	1	69	0	5
35	3	4	70	3	4
36	2	3	71	4	3
37	4	2	72	3	2
38	3	6	73	3	6
39	2	5	74	2	5
40	3	4	75	1	4
41	3	3	76	0	3
42	2	2	77	3	2
43	5	4	78	2	2
44	4	4	79	4	1
45	3	3	80	3	3
46	2	2	81	2	2
47	1	1	82	2	1
48	0	5	83	1	3
49	1	4	84	0	2
50	0	3	85	4	1
51	2	2	86	3	3
52	3	1	87	2	2
53	2	2	88	4	1
54	1	1	89	3	3
55	0	4	90	2	4
56	2	3	91	4	3
57	1	2	92	3	2
58	4	3	93	2	4
59	3	3	94	3	3
60	2	2	95	2	2
61	1	1	96	1	1
62	0	2	97	0	1
63	4	4	98	3	6
64	3	3	99	2	5
65	2	4	100	1	4

(continued)

k	$r(119, k)$	$r(127, k)$	k	$r(119, k)$	$r(127, k)$
101	0	3	116	1	2
102	2	2	117	1	1
103	1	1	118	0	4
104	0	3	119	3	4
105	2	2	120		3
106	4	1	121		3
107	3	2	122		2
108	2	2	123		3
109	2	1	124		2
110	1	1	125		1
111	0	2	126		0
112	3	1	127		2
113	3	3			
114	2	2			
115	2	1			