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ON THE STABILITY OF THE REPRODUCTIVE STRUCTURE OF A POPULATION

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Conditions have been obtained for the invariance of the proportions of pre-reproductive children, reproductive adults and post-reproductive old persons in a growing population. Normally the reproductive structure represented by these proportions changes with time, but it is shown here that there exists in general a unique reproductive structure which does not change with time. Even when the reproductive structure changes with time, it ultimately approaches the time-invariant structure. A method is also given for discussing the effects of changes in birth, death and internal migration parameters on the reproductive structure of the population.

1. BASIC SYSTEM OF DIFFERENTIAL EQUATIONS FOR OUR MODEL

We divide the population into three groups as follows:

(i) Children and young adolescents who cannot produce children but who on maturity and survival pass over into the second group of reproductive adults;

(ii) adults who can produce children and who on maturity and survival pass over into the third group of post-reproductive old persons;

(iii) old persons who do not produce children and who ultimately die.

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the number of persons in the three groups at time t. The ratios $x_1(t) : x_2(t) : x_3(t)$ will give the fertility or the reproductive structure of the population. If the ratios do not change with time, we shall say that the fertility structure of the population is time-invariant or that the population is 'structurally stable'.

Now in the time interval $(t, t + \triangle t)$, the following can happen:

(a) $x_1(t)$ can increase due to births of children and can decrease due to deaths among children and migration of children to adulthood. It is assumed that the number of births per unit time is proportional to the number of adults and the number of deaths and immigrations per unit time are proportional to the number of children. These considerations give

$$\frac{dx_1}{dt} = b_1 x_2 - d_1 x_1 - m_1 x_1 \qquad ...(1)$$

where b_1, d_1, m_1 are the birth, death and migration rate constants.

(b) $x_2(t)$ increases due to migration of children into adulthood and decreases due to deaths among adults and migration of adults into old age. This gives

$$\frac{dx_2}{dt} = m_1 x_1 - d_2 x_2 - m_2 x_2. \qquad ...(2)$$

(c) $x_3(t)$ increases due to migration of adults into old age and decreases due to deaths among old persons. This gives

$$\frac{dx_3}{dt} = m_2 x_2 - d_3 x_3. \qquad ...(3)$$

Adding (1), (2) and (3), we get

$$\frac{d}{dt}(x_1 + x_2 + x_3) = b_1 x_2 - d_1 x_1 - d_2 x_2 - d_3 x_3 \qquad ...(4)$$

which expresses the fact that the change in total population size per unit time is equal to the number of births per unit time minus the number of deaths per unit time in the three groups.

Equations (1), (2), (3) can be written in the matrix form

$$\frac{dX}{dt} = KX \qquad ...(5)$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad K = \begin{bmatrix} -(d_1 + m_1) & b_1 & 0 \\ m_1 & -(d_2 + m_2) & 0 \\ 0 & m_2 & -d_3 \end{bmatrix}. \qquad ...(6)$$

The nature of the solution of (5) will depend on the nature of the eigenvalues of the matrix K. We note that all the diagonal elements of the matrix K are negative and all the non-diagonal elements are non-negative. Further, the sums of elements of the first and third columns are negative and the sums of the elements of the second column will also be negative if $d_2 > b_1$ i.e. if the number of deaths per unit time among adults exceeds the number of births per unit time among them. In this case we have the result of Berman and Schoenfeld (1956) that the eigenvalues of K have negative real parts and purely imaginary eigenvalues are not possible. If $d_2 < b_1$, eigenvalues with non-negative real parts may arise.

In the present case, we can find the eigenvalues explicitly. Thus the first two eigenvalues λ_1, λ_2 are the roots of

$$f(\lambda) \equiv (\lambda + d_1 + m_1)(\lambda + d_2 + m_2) - b_1 m_1 = 0 \qquad ...(7)$$

and the third eigenvalues λ_3 is

$$\lambda_3 = -d. \qquad ...(8)$$

Equation (7) gives

$$\lambda_{1,2} = \tfrac{1}{2}\{-(d_1 + m_1 + d_2 + m_2) \pm \sqrt{(d_1 + m_1 + d_2 + m_2)^2 - 4[(d_1 + m_1)(d_2 + m_2) - b_1 m_1]}\} \qquad ...(9)$$

$$= \tfrac{1}{2}\{-(d_1 + m_1 + d_2 + m_2) \pm \sqrt{(d_1 + m_1 - d_2 - m_2)^2 + 4b_1 m_1}\}. \qquad ...(10)$$

The eigenvalues are all real and in general distinct. Two of them are negative and the third will be negative or positive according as

$$(d_1 + m_1)(d_2 + m_2) \gtrless b_1 m_1. \qquad ...(11)$$

We also note that

$$f(-\infty) > 0, f(-d_1 - m_1) < 0, f(-d_2 - m_2) < 0, f(\infty) > 0 \qquad ...(12)$$

so that λ_2 is less than both $-(d_1 + m_1)$ and $-(d_2 + m_2)$ and λ_1 is greater than both of them. A consequence of this would be that

$$\lambda_1 + d_1 + m_1 > 0, \lambda_1 + d_2 + m_2 > 0; \lambda_2 + d_1 + m_1 < 0,$$
$$\lambda_2 + d_2 + m_2 < 0. \qquad ...(13)$$

Since the three eigenvalues are in general distinct, we can write

$$K = Y \wedge Y^{-1} \qquad ...(14)$$

where $\wedge$ is the diagonal matrix of the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$, Y is the matrix whose column vectors are the right eigenvectors of the matrix K and Y^{-1} is the inverse of Y.

The differential eqn. (5) then becomes

$$\frac{dX}{dt} = Y \wedge Y^{-1} X. \qquad ...(15)$$

The solution of (15) is given by

$$X(t) = Y e^{\wedge t} Y^{-1} X(0) \qquad ...(16)$$

which gives

$$\left.\begin{aligned} x_1(t) &= c_{11} e^{\lambda_1 t} + c_{12} e^{\lambda_2 t} + c_{13} e^{\lambda_3 t} \\ x_2(t) &= c_{21} e^{\lambda_1 t} + c_{22} e^{\lambda_2 t} + c_{23} e^{\lambda_3 t} \\ x_3(t) &= c_{31} e^{\lambda_1 t} + c_{32} e^{\lambda_2 t} + c_{33} e^{\lambda_3 t}. \end{aligned}\right] \qquad ...(17)$$

From (11) and (17), we deduce the following:

(i) if $d_2 + m_2 > b_1$ i.e. if the number of deaths and migrations per unit time from the adult group exceeds the number of births per unit time, all the three eigenvalues are real and negative and all the three group populations will ultimately die out. This is also expected from common sense considerations.

(ii) Even if $d_2 > b_1$, the three eigenvalues are negative and the three group populations will die out. This result is consistent with the result deduced above from the theorem of Berman and Shcoenfeld (1956).

(iii) If $(d_2 + m_2) < b_1$ but $(d_1 + m_1)(d_2 + m_2) > b_1 m_1$, the three eigenvalues are still negative and the three group populations will still die out.

(iv) If $(d_1 + m_1)(d_2 + m_2) < b_1 m_1$, one of the eigenvalues is positive and the three group populations will not only survive but will in fact grow exponentially.

(v) If $(d_1 + m_1)(d_2 + m_2) = b_1 m_1$, one of the eigenvalues is zero and these populations can approach steady-state constant values.

2. The Ultimate Reproductive Structure of the Population

The matrix Y of the right eigenvectors of K is given by

$$Y = \begin{bmatrix} (d_3 + \lambda_1)(d_2 + m_2 + \lambda_1) & (d_3 + \lambda_2)(d_2 + m_2 + \lambda_2) & 0 \\ m_1(d_3 + \lambda_1) & m_1(d_3 + \lambda_2) & 0 \\ m_1 m_2 & m_1 m_2 & 1 \end{bmatrix}. \qquad ...(18)$$

Its inverse is given by

$$Y^{-1} = \frac{1}{\triangle} \begin{bmatrix} m_1(d_3 + \lambda_2) & -(d_2 + m_2 + \lambda_2)(d_3 + \lambda_2) & 0 \\ -m_1(d_3 + \lambda_1) & (d_2 + m_2 + \lambda_1)(d_3 + \lambda_1) & 0 \\ m_1^2 m_2(\lambda_1 - \lambda_2) & m_1 m_2(d_2 + m_2 + d_3 + \lambda_1 + \lambda_2)(\lambda_2 - \lambda_1) & \triangle \end{bmatrix} \qquad ...(19)$$

where

$$\triangle = m_1(d_3 + \lambda_1)(d_3 + \lambda_2)(\lambda_1 - \lambda_2) \qquad ...(20)$$

so that

$$e^{\Lambda t} Y^{-1} X(0) = \frac{1}{\triangle} \begin{bmatrix} e^{\lambda_1 t}(d_3 + \lambda_2)[m_1 x_1(0) - (d_2 + m_2 + \lambda_2) x_2(0)] \\ e^{\lambda_2 t}(d_3 + \lambda_1)[-m_1 x_1(0) + (d_2 + m_2 + \lambda_1) x_1(0)] \\ e^{\lambda_3 t}(\lambda_1 - \lambda_2)[m_1^2 m_2 x_1(0) - m_1 m_2(d_2 + m_2 + d_3 \\ + \lambda_1 + \lambda_2) x_2(0) + m_1(d_3 + \lambda_1)(d_3 + \lambda_2) x_3(0)] \end{bmatrix}. \qquad ...(21)$$

Since $X(t) = Y e^{\Lambda t} Y^{-1} X(0)$, we get

$$x_1(t) = \frac{1}{m_1(\lambda_1 - \lambda_2)}\{e^{\lambda_1 t}(d_2 + m_2 + \lambda_1)[m_1 x_1(0) - (d_2 + m_2 + \lambda_2)x_2(0)]$$
$$+ e^{\lambda_2 t}(d_2 + m_2 + \lambda_2)[-m_1 x_1(0) + (d_2 + m_2 + \lambda_1)x_2(0)]\} \qquad \ldots(22)$$

$$x_2(t) = \frac{1}{m_1(\lambda_1 - \lambda_2)}\{e^{\lambda_1 t} m_1[m_1 x_1(0) - (d_2 + m_2 + \lambda_2)x_2(0)]$$
$$+ e^{\lambda_1 t} m_1[-m_1 x_1(0) + (d_2 + m_2 + \lambda_1)x_2(0)]\} \qquad \ldots(23)$$

$$x_3(t) = \frac{1}{m_1(\lambda_1 - \lambda_2)} \cdot \Bigg\{ e^{\lambda_1 t} m_1 m_2 [m_1 x_1(0) - (d_2 + m_2 + \lambda_2)x_2(0)]/(d_3 + \lambda_1)$$
$$+ e^{\lambda_2 t} m_1 m_2[-m_1 x_1(0) + (d_2 + m_2 + \lambda_1)x_2(0)]/(d_3 + \lambda_2)$$
$$+ e^{\lambda_3 t}(\lambda_1 - \lambda_2)\frac{\begin{bmatrix} m_1^2 m_2 x_1(0) - m_1 m_2(d_2 + m_2 + d_3 + \lambda_1 \\ + \lambda_2)x_2(0) + m_1(d_3 + \lambda_1)(d_3 + \lambda_2)x_3(0) \end{bmatrix}}{(d_3 + \lambda_1)(d_2 + \lambda_2)}\Bigg\}. \qquad \ldots(24)$$

Now $\lambda_1 > \lambda_2$ and we assume $\lambda_2 < \lambda_3$.

As $t \to \infty$, the terms containing $e^{\lambda_1 t}$ dominate in (22), (23) and (24). Since from (13)

$$m_1 x_1(0) = (d_1 + m_2 + \lambda_2)x_2(0) \neq 0 \qquad \ldots(25)$$

then

$$\underset{t \to \infty}{\text{Lt}} \; x_1(t) : x_2(t) : x_3(t) = (d_3 + \lambda_1)(d_2 + m_2 + \lambda_1) : m_1(d_3 + \lambda_1) : m_1 m_2. \qquad \ldots(26)$$

Equation (26) gives the ultimate reproductive structure of the population. If however $\lambda_1 < 0$ and

$$d_3 < \tfrac{1}{2}[(d_1 + m_1 + d_2 + m_2) - \sqrt{(d_1 + m_1 - d_2 - m_2)^2 + 4b_1 m_1}] \qquad \ldots(27)$$

the terms containing $e^{\lambda_3 t}$ dominate. In this case the children and adult persons die out first and the older persons die out later. This case will however be rare and in general the ultimate structure will be given by (26).

3. Time-invariant Reproductive Structure

We want to investigate the possibility of a structure for which the ratios $x_1(t) : x_2(t) : x_3(t)$ do not depend on time or for which

$$x_1(t) : x_2(t) : x_3(t) = x_1(0) : x_2(0) : x_3(0) \qquad \ldots(28)$$

or for which

$$x_1(t) = x_1(0)f(t),\ x_2(t) = x_2(0)f(t),\ x_3(t) = x_3(0)f(t). \quad \text{...(29)}$$

Substituting in (1), (2), (3) we get

$$\frac{f'(t)}{f(t)} = b_1 \frac{x_2(0)}{x_1(0)} - (d_1 + m_1) = m_1 \frac{x_1(0)}{x_2(0)} - (d_2 + m_2)$$

$$= m_2 \frac{x_2(0)}{x_3(0)} - d_3 = c \text{ (say)}. \quad \text{...(30)}$$

(i) Eliminating $x_1(0)$, $x_2(0)$, $x_3(0)$, we get

$$(c + d_1 + m_1)(c + d_2 + m_2) = b_1 m_1 \quad \text{...(31)}$$

but this equation in c is the same as the eqn. (7) in λ so that

$$c = \lambda_1 \text{ or } \lambda_2. \quad \text{...(32)}$$

(ii) On integrating (30) and using $f(0) = 1$, we get

$$f(t) = e^{ct} = e^{\lambda_1 t} \text{ or } e^{\lambda_2 t}. \quad \text{...(33)}$$

(iii) From (30) and (32)

$$\frac{x_1(0)}{(\lambda_1 + d_2 + m_2)(\lambda_1 + d_3)} = \frac{x_2(0)}{m_1(\lambda_1 + d_3)} = \frac{x_3(0)}{m_1 m_2} \quad \text{...(34)}$$

or

$$\frac{x_1(0)}{(\lambda_2 + d_2 + m_2)(\lambda_2 + d_3)} = \frac{x_2(0)}{m_1(\lambda_2 + d_3)} = \frac{x_3(0)}{m_1 m_2}. \quad \text{...(35)}$$

However, since $\lambda_2 + d_2 + m_2 < 0$, the structure (35) is not possible.

(iv) If $\lambda_1 + d_3 < 0$, the first structure would also not be possible.

In this case no time-invariant structure exists.

(v) In practice however $\lambda_1 + d_3 > 0$ and in this case, we shall have for all time

$$x_1(t) : x_2(t) : x_3(t) = (\lambda_1 + d_2 + m_2)(\lambda_1 + d_3) : m_1(\lambda_1 + d_3) : m_1 m_2. \quad \text{...(36)}$$

Thus in general (i.e. when $\lambda_1 + d_3 > 0$) there is only one time-invariant structure given by (36). For this structure, the ratios of the three group populations do not change and these populations increase exponentially if $\lambda_1 > 0$ and decrease exponentially if $\lambda_1 < 0$.

There is of course a trivial time-invariant structure in which the three group populations are always zero.

4. Relation Between Ultimate Structure and Time-invariant Structure

We find that the ultimate structure given by (26) and the time-invariant structure (36) are the same.

The structure $x_1(t) : x_2(t) : x_3(t)$ can be represented by a point in the $u - v$ phase-space where

$$u = \frac{x_1(t)}{x_2(t)}, \; v = \frac{x_2(t)}{x_3(t)}. \qquad ...(37)$$

To every point in the positive quadrant of the phase space, there corresponds a structure and to every structure there corresponds a point in the positive quadrant of the phase space.

A given population has an initial structure $x_1(0) : x_2(0) : x_3(0)$ and as time increases, the structure goes on changing. To every growth of a population, there corresponds a curve in the phase space.

If the initial structure is given by (34), the curve reduces to a single point, as the structure does not change with time.

If the initial structure is different from (34), we get a curve in phase space. As t increases, the points on this curve approach the point corresponding to the time-invariant structure viz

$$\frac{d_2 + m_2 + \lambda_1}{m_1}, \frac{d_3 + \lambda_1}{m_2} \qquad ...(38)$$

The situation is represented in Fig. 1.

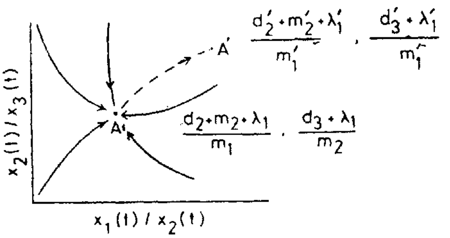

Fig. 1.

We may regard A as an absorbing state of the system. We may also regard the structure corresponding to A as a time-invariant 'stable' structure in the sense that if the population structure is disturbed, it will return to the original structure corresponding to A.

Thus if a population has been growing for a sufficiently long time according to our model, we should find it in state A. We should expect that the ratios of children to adults to old persons would not change with time and these ratios will give us two equations to estimate the parameters.

It is also clear that if the ratios are disturbed, they will again ultimately come to their original values. Thus during a war, there will be a significant number of deaths among young adults and the population structure will be disturbed, but some years after the war, the original structure will be re-established.

The structure will however be irreversibly changed if the birth, death or migration rates change on account of family planning measures or better health conditions or late marriages etc. If the parameters change from b_1, d_1, d_2, d_3, m_1, m_2 to b'_1, d'_1, d'_2, d'_3, m'_1, m'_2, the stable structure will change from A to A'.

The time in which the ultimate stable structure will be reached will be of the same order as the time it would take the terms containing $e^{\lambda_1 t}$ to dominate over the terms containing $e^{\lambda_2 t}$. We may take this time roughly as

$$\frac{4}{\lambda_1 - \lambda_2} = \frac{4}{\sqrt{(d_1 + m_1 - d_2 - m_2)^2 + 4b_1m_1}}. \qquad \ldots(39)$$

If $d_1 = 0.02$, $d_2 = 0.01$, $m_1 = 0.03$, $m_2 = 0.03$, $b_1 = 0.1$, this time is about 36 years. Thus a population with the above parameters will have reached the stable state in a period of at most 50 years and after that it will stay in that state and its structure will not change if the parameters do not change.

5. Effects of Changes on Parameters

It is obvious that a sufficiently good idea of these effects can be obtained by considering these effects on the ultimate stable structure, unless changes in parameters take place in times of much smaller order than the time taken to reach a stable structure.

The ultimate structure is given by

$$(d_2 + m_2 + \lambda_1)(d_3 + \lambda_1) : m_1(d_3 + \lambda_1) : m_1m_2 \qquad \ldots(40)$$

where

$$\lambda_1 = \tfrac{1}{2}[-(d_1 + m_1 + d_2 + m_2) + \sqrt{(d_1 + m_1 - d_2 - m_2)^2 + 4b_1m_1}]. \qquad \ldots(41)$$

(i) If b_1 decreases, λ_1 decreases and therefore the population of children decreases relative to that of adults and the population of adults decreases relative to that of old persons. As such the decrease of birth rate tends to change the population structure in favour of the older persons. This effect has already been noticed in those developed countries where the birth rates have appreciably declined. Lowering birth rates tends to increase the proportion of older people and raising birth rate tends to increase the proportion of children and adults.

(ii) If d_3 decreases i.e. if the death rate among old persons decreases due possibly to better health care for them, the population of children does not change relative to that of adults, but the population of both groups decrease relative to that of the old persons. This is of course expected but our result gives a quantitative estimate for this decrease.

(iii) We cannot make similar general statements for changes in the other four parameters d_1, d_2, m_1, m_2 but given the present values of d_1, m_1, d_2, m_2 and given the changes in these, we can easily find the changes in the structure of the population by using (40) and (41). For this purpose it is essential that we should be able to estimate the values of these parameters from given data about the growth of the population.

6. Estimation of the Values of the Parameters

We know that $x_1(t)$, $x_2(t)$, $x_3(t)$ are given by (17). If we have data for $x_1(t)$, $x_2(t)$, $x_3(t)$ for a sufficient number of values of t, we can fit curves of the form (17) to the given data by the method of peeling of exponentials (Bernardelli 1941). This will enable us to estimate

$$\lambda_1, \lambda_2, \lambda_3; c_{11}, c_{12}, c_{13}; c_{21}, c_{22}, c_{23}; c_{31}, c_{32}, c_{33} \qquad \ldots(42)$$

from which by using (8), (9), (10), (22), (23), (24), we can estimate

$$b_1, d_1, d_2, d_3; m_1, m_2; x_1(0), x_2(0), x_3(0). \qquad \ldots(43)$$

We can get additional confirmation of our estimates by using (40), (41) for the stable population structure.

7. Comparison with Earlier Stable Population Theory

Our present model which is a continuous-time discrete-age-scale population model can easily be extended to the case when the population has n age-groups, of which first p are pre-reproductive, next q are productive and the last r are post-productive where $p + q + r = n$. For this model, the basic matrix K is an $n \times n$ matrix, whose diagonal elements are $-(d_i + m_i)$, $(i = 1, 2, \ldots, n)$, whose main subdiagonal elements are m_i $(i = 1, 2, \ldots, n-1)$, whose $(p+1)$th to $(p+q)$th elements of the first row are $b_{p+1}, b_{p+2}, \ldots, b_{p+q}$ and the rest of whose elements are zero. The stable age-structure is given by

$$\frac{x_1}{(d_2 + m_2 + \lambda_0) \ldots (d_n + m_n + \lambda_0)} = \frac{x_2}{m_1(d_3 + m_3 + \lambda_0) \ldots (d_n + m_n + \lambda_0)} = \ldots \frac{x_n}{m_1 m_2 \ldots m_{n-1}} \qquad \ldots(44)$$

where λ_0 is the dominant eigenvalue of the matrix K.

A second alternative age-structured population model is the 'discrete-time discrete-age-scale' population model given by Bernardelli (1941), Lewis (1942), and Leslie (1945, 1948). The stable population theory for it has been given in Keyfitz (1968), Pielou (1969), Usher (1972), Pollard (1973) and Kapur (1978b). Pollard (1966, 1973) has also briefly discussed the effects of immigration and emigration.

The main differences between our model and that of Leslie arise due to the following reasons:

(i) Our model is a continuous-time model and it leads to a system of differential equations, while Leslie's model is a discrete-time model and leads to a system of difference equations. Even for linear models, the difference is important, but for non-linear density-dependent models (Kapur 1979a), the difference equations models lead to serious mathematical and conceptual difficulties as pointed out by Li and Yorke (1979), Oster (1976) and Kapur (1979b).

(ii) In our model, it is not necessary that different age-groups should cover equal lengths of time intervals, but this is essential for the model of Leslie and this is a serious restriction on that model.

(iii) We can also convert our differential equation system to a difference equation system, but the converted system will have an advantage over Leslie's system in the sense that different age-groups can cover different time intervals and the interval of differencing can be different from each of them. In the case of Leslie's model, the interval of differencing is the same as that for each age group.

(iv) Leslie's matrix is non-negative and the well-developed theory for non-negative matrices due to Brauer (1962) and Sykes (1969) can be applied to it. For our model, all the diagonal elements of the matrix K are negative and the theory for such matrices is still being developed.

(v) The theory for Leslie's model has been developed during nearly four decades by a large number of workers. The corresponding results for our model are being developed, taking into account the special nature of the matrix K. Thus Lopez (1961) has proved the weak ergodicity property viz. that two populations which are arbitrarily different in their age structure will tend to adopt the same age distribution as each other with the passage of time if they are both subjected to the same sequence of fertility and mortality rates which are not assumed to be unchanging. The corresponding result can also be proved for our model. Pollard's (1966) stochastic version of Leslie's model has however been already developed for the continuous-time model (Kapur 1978a).

A third type of model is the continuous-time continuous-age-scale of Sharpe and Lotka (1911) developed further by Rhodes (1940), Feller (1941) and Keyfitz (1968). This is based on the solution of integral equations. This also gives a stable age distribution. Another model of the same type has been given by Gurtin and

Maccamy (1974). These models are mathematically more sophisticated but do not give results in the same details as the Leslie-type model.

Our model lies somewhere between the models of Leslie and Sharpe and Lotka in both mathematical elegance and sophistication and in details of result.

The three types of models have different roles in spite of the fact that they may give many similar results.

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