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INVERSE OF A TRIDIAGONAL BAND MATRIX IN TERMS OF GENERALIZED HERMITE POLYNOMIALS

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The purpose of this paper is to derive inverse of a tridiagonal band matrix in terms of the generalized Hermite polynomials introduced by Gould and Hopper (1962).

1. INTRODUCTION

Vimuri (1969) obtained inverse of a tridiagonal band matrix in terms of Gegenbauer polynomial $C_n^{\alpha}(x)$ for $\alpha = 1$. In the present paper, we have obtained inverse of a tridiagonal band matrix in terms of the generalized Hermite polynomials $g_n^m(x, \lambda)$ introduced by Gould and Hopper (1962). These polynomials include, as special cases, the Hermite polynomials $H_n(x)$ (1864), Widder's polynomials (1975), and the polynomials studied by Lahiri (1971). Recently, these polynomials have been further investigated by Rekha (1976). In fact, Rekha studied the polynomials $H_{n,m,\nu}(x, \lambda)$ which are Gould and Hopper's polynomials with x replaced by νx; i.e., $H_{n,m,\nu}(x, \lambda) = g_n^m(\nu x, \lambda)$. The polynomials $g_n^m(x, \lambda)$ are defined by

$$g_n^m(x, \lambda) = \sum_{k=0}^{[n/m]} \frac{n!}{k!\,(n-mk)!} \lambda^k x^{n-mk}. \qquad \ldots(1.1)$$

When $m = 2$, $\lambda = -1$ and x is replaced by $2x$ then (1.1) reduces to Hermite polynomials $H_n(x)$ (1864). If we take $m = 2$ in (1.1), we get Widder's polynomials $V_n(x, \lambda)$ (1975). Again if $\lambda = -1$ and x is replaced by νx, then $g_n^m(x, \lambda)$ reduces to the polynomial studied by Lahiri (1971).

2. THE TRIDIAGONAL BAND MATRIX AND ITS INVERSE

Consider a square matrix $A_n = [a_{i,j}]$ of order n, such that

$$\left.\begin{array}{lll} a_{i,j} = 0 & \text{when} & j \neq i-1, i, i+1 \\ a_{i,j} = 2i-2 & \text{when} & j = i-1 \\ a_{i,j} = x & \text{when} & j = i \\ \text{and} \quad a_{i,j} = \lambda & \text{when} & j = i+1 \end{array}\right\} \qquad \ldots(2.1)$$

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so that

$$A_n = [a_{i,j}] = \begin{bmatrix} x & \lambda & 0 & 0 & \dots & 0 & 0 \\ 2 & x & \lambda & 0 & \dots & 0 & 0 \\ 0 & 4 & x & \lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x & \lambda \\ 0 & 0 & 0 & 0 & \dots & 2n-2 & x \end{bmatrix}. \qquad \dots(2.2)$$

Let $D_n = D_n(x)$ denote the determinant of $[a_{i,j}]$. Then

$$\left.\begin{aligned} D_0(x) &= 1 \\ D_1(x) &= x \\ D_2(x) &= x^2 - 2\lambda \\ D_3(x) &= x^3 - 6\lambda x \\ D_4(x) &= x^4 - 12\lambda x^2 + 12\lambda^2. \end{aligned}\right\} \qquad \dots(2.3)$$

In general

$$D_n(x) = \sum_{k=0}^{[n/2]} \frac{(-\lambda)^k \, n!}{k! \, (n-2k)!} \, x^{n-2k} \qquad \dots(2.4)$$

which by (1.1) is nothing but $g_n^2 \, (x, -\lambda)$.

For simplicity we shall consider the case when $n = 4$. Now the inverse of the matrix $[a_{i,j}]_{4\times 4}$ in (2.2) can be obtained by elementary matrix algebra as

$$A_4^{-1} = \frac{1}{\{x^4 - 12\lambda x^2 + 12\lambda^2\}} \times \begin{bmatrix} x^3 - 10\lambda x & -\lambda x^2 + 6\lambda^2 & \lambda^2 x & -\lambda^3 \\ -2x^2 + 12\lambda & x^3 - 6\lambda x & -\lambda x^2 & \lambda^2 x \\ 8x & -4x^2 & x^3 - 2\lambda x & -\lambda x^2 + 2\lambda^2 \\ -48 & 24x & -6x^2 + 12\lambda & x^3 - 6\lambda x \end{bmatrix}. \qquad \dots(2.5)$$

On using (2.3), we get

$$A_4^{-1} = \frac{1}{D_4} \begin{bmatrix} D_3 - 4\lambda D_1 & -\lambda D_2 + 4\lambda^2 D_0 & \lambda^2 D_1 & -\lambda^3 D_0 \\ -2D_2 + 8\lambda D_0 & D_3 & -\lambda D_2 - 2\lambda^2 D_0 & \lambda^2 D_1 \\ 8D_1 & -4D_2 - 8\lambda D_0 & D_3 + 4\lambda D_1 & -\lambda D_2 \\ -48D_0 & 24D_1 & -6D_2 & D_3 \end{bmatrix} \qquad \dots(2.6)$$

which, can easily be transformed to

$$A_4^{-1} = \frac{1}{g_4^2}\begin{bmatrix} g_3^2 - 4\lambda g_1^2 & -\lambda g_2^2 + 4\lambda^2 g_0^2 & \lambda^2 g_1^2 & -\lambda^3 g_0^2 \\ -2g_2^2 - 8\lambda g_0^2 & g_3^2 & -g_2^2 - 2\lambda^2 g_0^2 & \lambda^2 g_1^2 \\ 8g_1^2 & -4g_2^2 - 8\lambda g_0^2 & g_3^2 + 4\lambda g_1^2 & -\lambda g_2^2 \\ -48g_0^2 & 24g_1^2 & -6g_2^2 & g_3^2 \end{bmatrix} \quad \ldots(2.7)$$

where, of course, g_n^2 means $g_n^2\,(x, -\lambda)$. Similarly, for any value of n, we can find inverse of matrix A_n in terms of the polynomials $g_n^2\,(x, -\lambda)$.

In general, if $[a_{i,j}]$ be an $n \times n$ matrix such that D_n, the determinant of $[a_{i,j}]$ is equal to a polynomial, $P_n(x)$ for each n, then the inverse of $[a_{i,j}]$ can be obtained in terms of the polynomials $P_i(x)$ for $i = 0, 1, 2, \ldots, n$.

If we replace x by $2x$ and put $\lambda = 1$ in (2.2), then we get (2.4) as

$$D_n(2x) = \sum_{k=0}^{[n/2]} \frac{n!\,(-1)^k}{k!\,(n-2k)!}\,(2x)^{n-2k} \quad \ldots(2.8)$$

where $D_n(2x)$ is the well-known Hermite polynomial $H_n(x)$ and we obtain inverse of

$$[a_{i,j}]_{4\times 4} = \begin{bmatrix} 2x & 1 & 0 & 0 \\ 2 & 2x & 1 & 0 \\ 0 & 4 & 2x & 1 \\ 0 & 0 & 6 & 2x \end{bmatrix} \quad \ldots(2.9)$$

as

$$\frac{1}{H_4}\begin{bmatrix} H_3 - 4H_1 & -H_2 + 4H_0 & H_1 & -H_0 \\ -2H_2 + 8H_0 & H_3 & -H_2 - 2H_0 & H_1 \\ 8H_1 & -4H_2 - 8H_0 & H_3 + 4H_1 & -H_2 \\ -48H_0 & 24H_1 & -6H_2 & H_3 \end{bmatrix}. \quad \ldots(2.10)$$

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References

Gould, H. W., and Hopper, A. T. (1962). Operational formulas connected with two generalizations of Hermite polynomials. *Duke Math. J.*, **29**, 51-63.

Hermite, Ch. (1864). Sur un noureau développements on serie de Fonctions. *Compt. Rend. Acad. Sci. (Paris)*, **58**, 93-100, 266-273.

Lahiri, M. (1971). On a generalization of Hermite polynomials. *Proc. Am. math. Soc.*, **27**, 117-21.

Rekha, Suman (1976). A study of some polynomial sets. Ph.D. thesis, University of Delhi, Delhi.

Vimuri, V. (1969). A novel way of expressing a class of tridiagonal band matrix in terms of Gegenbauer polynomial. *Matrix & Tensors Quart.*, **20**, 55-58.

Widder, D. V. (1975). The Heat Equation. Academic Press, New York.