Indian J. pure appl. Math., **10**(10) : 1292-1297, October 1979

ON A SUBCLASS OF SPIRAL-LIKE FUNCTIONS

PRAKASH G. UMARANI

Department of Mathematics, S. Nijalingappa College, Rajaji Nagar, Bangalore 560010

(*Received* 19 *July* 1978; *after revision* 16 *February* 1979)

The author introduces a subclass of spiral-like functions which for different values of α and β leads to the class of α-convex functions and spiral-convex functions. Further he obtains a coefficient inequality for this class.

§1. Let $f(z) = z + a_2z^2 + \ldots$, be analytic in the unit disk E with

$$\frac{f(z)\,f'(z)}{z} \neq 0$$

in E, and let α be a real number. Then $f(z)$ is said to be α-convex in E if and only if the inequality

$$\text{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0 \qquad \ldots(1.1)$$

holds in E.

This class of functions was introduced by Mocanu (1969). Miller *et al.* (1973) have proved that α-convex functions are univalent and starlike. Denote the class of α-convex functions by S_α.

A function $f(z) = z + a_2z^2 + \ldots$, regular in E is said to be spiral-covex if and only if the inequality

$$\text{Re}\left\{\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right) + i\sin\beta\,\frac{zf'(z)}{f(z)}\right\} > 0 \qquad \ldots(1.2)$$

holds in E, for some $\beta : -\pi/2 < \beta < \pi/2$. Denote this class by SC_β. Yoshikawa (1971) introduced the class SC_β and showed that the functions of SC_β are spiral-like.

In this paper a subclass of spiral-like functions, which for different values of α and β leads to the class of α-convex functions and spiral-convex functions has been introduced. Further the problem of maximizing $|\ a_3 - \mu a_2^2\ |$ for any complex number μ over this class has been considered.

Definition — Let $f(z) = z + a_2z^2 + \ldots$, be regular in E with $\frac{f(z)f'(z)}{z} \neq 0$ in E and α be a real number. Then $f(z)$ is said to be α-spiral-convex function if and only if it satisfies the inequality.

$$\operatorname{Re}\left\{(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0$$

in E, for some fixed $\beta : -\pi/2 < \beta < \pi/2$. Denote this class by $SC(\alpha, \beta)$.

Clearly $SC(\alpha, 0)$ is the class of α-convex functions and $SC(1, \beta)$ is the class of spiral-convex functions and $SC(0, \beta)$ is the class of β-spiral functions.

§2. The following Lemma due to Jack (1971) will be used in the proof of Theorem 1.

Lemma 1 — Let $w(z)$ be regular in E with $w(0) = 0$. If there exists a $\xi \in E$ such that

$$\underset{|z| \leqslant |\xi|}{\operatorname{Max}} \; w(z) = |w(\xi)|$$

then

$$\xi w'(\xi) = k \,.\, w(\xi) \text{ for some } k \geqslant 1.$$

Theorem 1 — If $f(z) \in SC(\alpha, \beta)$, then $f(z)$ is β-spiral.

PROOF : Let

$$e^{i\beta}\frac{zf'(z)}{f(z)} = \cos\beta\,\frac{1 - w(z)}{1 + w(z)} + i\sin\beta,\ z \in E. \qquad \ldots(2.1)$$

This implies that

$$\frac{zf'(z)}{f(z)} = \frac{1 - e^{-2i\beta}w(z)}{1 + w(z)}.$$

Therefore, $w(z) \neq -1$ and $w(z) \neq e^{2i\beta}$. Clearly $w(0) = 0$. Now to prove $f(z)$ is β-spiral, it is sufficient to show that $|w(z)| < 1$ in E.

Suppose that $\underset{|z| \leqslant |\xi|}{\operatorname{Max}} |w(z)| = |w(\xi)| = 1$ for some $\xi \in E$.

Then by the Lemma 1, $\xi\omega'(\xi) = k\omega(\xi)$ for some $k \geqslant 1$.

If we show that

$$\operatorname{Re}\left\{(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} = 0 \qquad \ldots(2.2)$$

at the point $z = \xi$, then it contradicts the fact that $f(z) \in SC(\alpha, \beta)$ and thus our theorem will be proved.

From (2.1), we get

$$\text{Re}\left\{(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\}$$

$$= \text{Re}\left\{\cos\beta \,.\, p(z) + i\sin\beta + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right\}$$

$$= \text{Re}\left\{\cos\beta p(z) + i\sin\beta + \alpha\cos^2\beta\frac{zp'(z)}{\cos\beta p(z) + i\sin\beta}\right\} \quad \ldots(2.3)$$

where $p(z) = \dfrac{1 - w(z)}{1 + w(z)}$.

Now it easily follows that $p(\xi)$ is imaginary and $\xi p'(\xi)$ is purely real. Hence from (2.3) we have

$$\text{Re}\left\{(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} = 0 \text{ at } z = \xi.$$

This completes the proof of the theorem.

Theorem 2 — $f(z) \in SC(\alpha, \beta)$ if and only if there exists a β-spiral function $g(z)$ such that

$$g(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^{\alpha\cos\beta e^{-i\beta}} \quad \ldots(2.4)$$

where the branch of $\left[\dfrac{zf'(z)}{f(z)}\right]^{\alpha\cos\beta e^{-i\beta}}$ is chosen equal to 1 at $z = 0$.

PROOF : Let $f(z) \in SC(\alpha, \beta)$. Then by the well-known Herglotz representation, we have

$$(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$= \cos\beta\int_0^{2\pi}\frac{1 + ze^{-it}}{1 - ze^{-it}}\,d\mu(t) + i\sin\beta$$

where $\mu(t)$ is a non-decreasing function with $\int_0^{2\pi} d\mu(t) = 1$.

A little computation and an integration gives us

$$\log\frac{f(z)}{z}\left[\frac{zf'(z)}{f(z)}\right]^{\alpha\cos\beta e^{-i\beta}} = -2\cos\beta e^{-i\beta}\int_0^{2\pi}\log(1 - ze^{-it})\,d\mu(t).$$

Now there exists some β-spiral function $g(z)$ such that the right-hand side of the above is equal to $\log g(z)/z$.

Hence (2.4) follows.

Conversly if $g(z)$ given by (2.4) is β-spiral, then we have

$$\operatorname{Re}\left\{(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} = \operatorname{Re}\frac{e^{i\beta}zg'(z)}{g(z)} > 0.$$

$f(z) \in SC(\alpha, \beta)$.

This completes the proof of the theorem.

§3. *Coefficient Inequality for* $SC(\alpha, \beta)$ — The following lemma due to Keogh and Merkes (1969) will be used to prove Theorem 3.

Lemma 2 — Let $w(z) = c_1z + c_2z^2 + \ldots$, be analytic with $|w(z)| < 1$ in E. If υ is any complex number, then

$$|c_2 - \upsilon c_1^2| \leqslant \operatorname{Max}\{1, |\upsilon|\}. \qquad \ldots(3.1)$$

Equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

Theorem 2 — Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$, belong to $SC(\alpha, \beta)$ and μ be any complex number.

Then

$$|a_3 - \mu a_2^2| \leqslant \frac{\cos\beta}{|e^{i\beta} + 2\alpha\cos\beta|}\operatorname{Max}\{1, |\upsilon|\} \qquad \ldots(3.2)$$

where

$$\upsilon = \frac{4\cos\beta(e^{i\beta} + 2\alpha\cos\beta)\,\mu - [2\cos\beta(e^{i\beta} + 3\alpha\cos\beta) + (e^{i\beta} + \alpha\cos\beta)^2]}{(e^{i\beta} + \alpha\cos\beta)^2}$$

The result is sharp.

Proof : Let $f(z) \in SC(\alpha, \beta)$, then there exists an analytic function

$$w(z) = c_1z + c_2z^2 + \ldots, \text{ with } |w(z)| < 1$$

in E such that

$$(e^{i\beta} - \alpha\cos\beta)\frac{zf'(z)}{f(z)} + \alpha\cos\beta\left(1 + \frac{zf''(z)}{f'(z)}\right)$$

$$= \cos\beta\,\frac{1 + w(z)}{1 - w(z)} + i\sin\beta$$

$$= \frac{e^{i\beta} + e^{-i\beta}w(z)}{1 - w(z)}. \qquad \ldots(3.3)$$

Equating the coefficients in (3.3) we get

$$c_1 = \frac{e^{i\beta} + \alpha \cos \beta}{2 \cos \beta} a_2$$

$$c_2 = \frac{e^{i\beta} + 2\alpha \cos \beta}{\cos \beta} a_3 - \frac{[2 \cos \beta(e^{i\beta} + 3\alpha \cos \beta) + (e^{i\beta} + \alpha \cos \beta)^2]}{4 \cos^2 \beta} a_2^2$$

$$c_2 - \upsilon c_1^2 = \frac{e^{i\beta} + 2\alpha \cos \beta}{\cos \beta} \{a_3 - \mu a_2^2\}$$

where

$$\mu = \frac{\cos \beta}{(e^{i\beta} + 2\alpha \cos \beta)} \left\{ \frac{[2 \cos \beta(e^{i\beta} + 3\alpha \cos \beta) + (e^{i\beta} + \alpha \cos \beta)^2]}{4 \cos^2 \beta} + \frac{\upsilon(e^{i\beta} + \alpha \cos \beta)^2}{4 \cos^2 \beta} \right\}$$

From Lemma 2 and the above, theorem follows.

The equality in (3.2) can be attained for a function $f(z)$ which satisfies

$$f(z) \left[\frac{zf'(z)}{f(z)}\right]^{\alpha \cos \beta e^{-i\beta}} = \frac{z}{(1 - z)^2 \cos \beta e^{-i\beta}}.$$

If $\alpha = 1$, we get a new result for spiral-convex functions.

Corollary 1 — If $f(z) = z + a_2 z^2 + \ldots,$ is spiral-convex and μ is any complex number, then

$$| a_3 - \mu a_2^2 | \leqslant \frac{\cos \beta}{\sqrt{1 + 8 \cos^2 \beta}} \text{Max} \{1, | \upsilon | \}$$

where

$$\upsilon = \frac{4 \cos \beta(e^{i\beta} + 2 \cos \beta) \mu - [2 \cos \beta(e^{i\beta} + 3 \cos \beta) + (e^{i\beta} + \cos \beta)^2]}{(e^{i\beta} + \cos \beta)^2}$$

If $\beta = 0$, we get the coefficient inequality for α-convex functions which is due to Szynal (1972).

Corollary 2 — If $f(z) = z + a_2 z^2 + \ldots,$ is α-convex and μ is any complex number, then

$$| a_3 - \mu a_2^2 | \leqslant \frac{1}{1 + 2\alpha} \text{Max} \{1, | \upsilon | \}$$

where

$$\upsilon = \frac{4\mu(1 + 2\alpha) - [2(1 + 3\alpha) + (1 + \alpha)^2]}{(1 + \alpha)^2}.$$

If $\alpha = 0$, we get a coefficient inequality for β-spiral functions which is due to Keogh and Merkes (1969).

ACKNOWLEDGEMENT

The author wishes to thank the referee for his valuable suggestions.

REFERENCES

Jack, I. S. (1971). Functions starlike and convex of order α. *J. Lond. math. Soc.* (2), **3**, 469-74.

Keogh, F. R., and Merkes, E. P. (1969). A coefficient inequality for certain classes of analytic functions. *Proc. Am. math. Soc.*, **20**, 8-12.

Miller, S. S., Mocanu, P. T., and Reade, M. O. (1973). All α-convex functions are univalent and starlike. *Proc. Am. math. Soc.*, **37**, 553-54.

Mocanu, P. T. (1969). Une propriété de convexité generalisée dans la théorie de la représentation conforme. *Mathematica (Cluj)*, **11**(34) 127-33.

Szynal, J. (1972). Some remarks on coefficient inequality for α-convex functions. *Bull. Acad. Pol. Sci.*, **20**, 917-19.

Yoshikawa, H. (1971). On a subclass of spiral-like functions. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, **25**, 271-79.