Indian J. pure appl. Math., **10**(10) : 1298-1302, October 1979

ON THE UNIFORM NÖRLUND SUMMABILITY OF LEGENDRE SERIES

RAJENDRA PRASAD

Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi 221005

(*Received* 12 *July* 1978; *after revision* 17 *April* 1979)

In this paper two theorems on the uniform Nörlund summability of Legendre series, which are analogous to the theorems of Saxena (1965) for Fourier series, are established.

§1. The Legendre series associated with the Lebesgue-integrable function $f(x)$ in the range $(-1, 1)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \qquad \text{...(1.1)}$$

where

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} f(t)\, P_n(t)\, dt \qquad \text{...(1.2)}$$

and the Legendre polynomials $P_n(x)$, which are orthogonal in the interval $(-1, 1)$, are defined by the generating function

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \qquad \text{...(1.3)}$$

However, if the coefficient a_n's are not restricted by the relation (1.2) then the series (1.1) is known as series of Legendre polynomial.

Saxena (1965) introduced the concept uniform Nörlund summability, which is as follows :

Let

$$u_0(x) + u_1(x) + \ldots \qquad \text{...(1.4)}$$

be any infinite series and define

$$U_\nu(x) = u_0(x) + u_1(x) + \ldots + u_\nu(x).$$

Let $\{P_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n$$

If there exists a function $U = U(x)$ such that.

$$\frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \{U_{n-\nu}(x) - U(x)\} = o(1) \qquad \ldots(1.5)$$

uniformly in a set E in which $U = U(x)$ is bounded, then the series (1.4) is summable (N, p_n) uniformly in E to the sum U.

§2. Tripathi (1977) has studied the Nörlund summability of Legendre series. The object of the present paper is to establish the following theorems on the uniform Nörlund summability of Legendre series :

Theorem 1 — If $\alpha(t)$ denotes a function of t, $\alpha(t)$ and $t/\alpha(t)$ ultimately increase steadily with t,

$$\int_0^t | f(x \pm u) - f(x) | \, du = o\left(\frac{t}{\alpha(P_\tau)}\right) \qquad \ldots(2.1)$$

uniformly in a set E defined in the interval $(-1, 1)$, in which $f(x)$ is bounded as $t \to +0$ then the series (1.1) is summable (N, p_n) uniformly in E to the sum $f(x)$, where $\{p_n\}$ is real non-negative monotonic non-increasing sequence such that $P_n \to \infty$ as $n \to \infty$, provided that

$$\log n = O(\alpha(P_n)) \text{ as } n \to \infty \text{ holds.} \qquad \ldots(2.2)$$

where $\tau = [1/t]$, $[\lambda]$ denotes the integral part of λ.

Theorem 2 — If $\alpha(t)$ stands for a function of t and ultimately increases steadily with t

$$\int_0 | f(x \pm u) - f(x) | \, du = o\left(\frac{t}{\alpha(P_\tau)}\right) \qquad \ldots(2.3)$$

uniformly in a set E defined in the interval $(-1, 1)$ in which $f(x)$ is bounded as $t \to +0$, then the series (1.1) is summable (N, p_n) uniformly in E to the sum $f(x)$ where $\{p_n\}$ is defined in Theorem 1, provided that

$$\int_{1/n}^{\eta} \frac{P_\tau}{\alpha(P_\tau)} \cdot \frac{1}{t} \, dt = O(P_n) \text{ as } n \to \infty, \text{ holds.} \qquad \ldots(2.4)$$

§3. *Proof of Theorem* 1 — The nth partial sum of the series (1.1) is given by

$$S_n(x) - f(x) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} \frac{f(\cos(\theta - t)) - f(\cos\theta)}{\sin^{1/2} t}$$

$$\times \sin(n + 1)t \sin^{1/2}(\theta - t)\, dt + o(1)$$

where $0 < \eta \leqslant \delta < 1$, $x = \cos\theta$, $y = \cos\phi$, $0 < \theta < \pi$, $\theta - \phi = t$, $0 < \phi < \pi$ etc.

Since $f(x)$ is bounded on the set E. So $o(1)$ will tend to zero for any x uniformly in E.

Now

$$\frac{1}{P_n} \sum_{k=0}^{n} p_k\{S_{n-k}(x) - f(x)\}$$

$$= \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\eta} [f(\cos(\theta - t)) - f(\cos\theta)] \sin^{1/2}(\theta - t)$$

$$\times \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{\sin(n - k + 1)t}{\sin^{1/2} t}\, dt$$

$$+ o(1) \text{ uniformly in } E$$

$$= O\left(\left\{\int_0^{1/n} + \int_{1/n}^{\eta}\right\} |\psi(t)|\; |N_n(t)|\, dt\right) + o(1) \text{ uniformly in } E$$

$$= O(I_1) + O(I_2) + o(1) \text{ uniformly in } E \text{ (say)} \qquad \ldots(3.1)$$

where $\psi(t) = f(\cos(\theta - t)) - f(\cos\theta)$

and $N_n(t) = \dfrac{1}{P_n} \displaystyle\sum_{k=0}^{n} p_k \frac{\sin(n - k + 1)t}{\sin^{1/2} t}$.

In order to prove our theorem we have to show that under our assumption

$I_1 = o(1)$ and $I_2 = o(1)$ as $n \to \infty$, uniformly in the set E.

Now uniformly in $0 < t \leqslant \dfrac{1}{n}$

$N_n(t) = O(n)$

So,

$$I_1 = \int_0^{1/n} |\psi(t)| \; |N_n(t)| \, dt$$

$$= O\left(n \, o\left(\frac{n-1}{\alpha(P_n)}\right)\right) \text{ uniformly in } E$$

$$= o(1) \text{ uniformly in } E \text{ as } n \to \infty, \text{ by hypothesis.} \qquad \ldots(3.2)$$

Again, since $\frac{1}{n} \leqslant t \leqslant \eta < \pi$, $N_n(t) = O\left(\frac{P_\tau}{t\,P_n}\right)$

Therefore,

$$I_2 = O\left(\int_{1/n}^{\eta} |\psi(t)| \; \frac{P_\tau}{t\,P_n} \, dt \right)$$

$$= O\left(\frac{1}{P_n} \int_{1/n}^{\eta} |\psi(t)| \cdot \frac{P_\tau}{t} \, dt \right)$$

$$= O\left(\frac{1}{P_n} \left[\Psi(t) \cdot \frac{P_\tau}{t} \right]_{1/2}^{\eta} + \frac{1}{P_n} \int_{1/n}^{\eta} \Psi(t) \, \frac{P_\tau}{t^2} \, dt \right)$$

$$+ \; o(1) \text{ uniformly in } E$$

$$= \left\{ o\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(P_n)}\right) + o\left(\frac{1}{P_n} \int_{1/n}^{\eta} \frac{P_\tau}{\alpha(P_\tau)} \cdot \frac{1}{t} \, dt\right) + o(1) \right\}$$

uniformly in E

$$= \left\{ o\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(P_n)}\right) + o\left(\frac{1}{\alpha(P_n)} \int_{1/n}^{\eta} \frac{1}{t} \, dt\right) + o(1) \right\}$$

uniformly in E (by the hypothesis of the theorem)

$$= \left\{ o\left(\frac{1}{P_n}\right) + o\left(\frac{1}{\alpha(P_n)}\right) + o\left(\frac{\log n}{\alpha(P_n)}\right) + o(1) \right\} \text{ uniformly in } E$$

$$= \{o(1) + o(1) + o(1) + o(1)\} \text{ uniformly in } E \text{ as } n \to \infty$$

(by the hypothesis of the theorem)

$$= o(1) \text{ uniformly in } E. \qquad \ldots(3.3)$$

Now, from (3.1), (3.2) and (3.3) Theorem 1 follows.

Proof of Theorem 2 — The proof of Theorem 2 runs parallel to that of Theorem 1.

ACKNOWLEDGEMENT

The author is very much thankful to the referee for his valuable suggestions and generous encouragement for the preparation of this paper.

REFERENCES

Saxena, A. (1965). On uniform Nörlund summability of Fourier series. *Proc. natn. Inst. Sci. India*, A **32**, 502-508.

Tripathi, L. M. (1977). On the Nörlund summability of Legendre series. *Prog. Math.* (*Allahabad*), **11**, 85-91.