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ON THE NÖRLUND SUMMABILITY OF FOURIER-JACOBI SERIES

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In this paper we have obtained a result on the (N, p_n) summability of Fourier-Jacobi series. The result obtained extends the theorem of Sharma (1976) on the Nörlund summability of Fourier-Jacobi series.

§1. The Jacobi-polynomial $P_n^{(\alpha,\beta)}(x)$, $\alpha > -1$, $\beta > -1$ are defined by

$$2^{\alpha+\beta}(1-2xt+t^2)^{-1/2}\,[1-t+(1-2xt+t^2)^{1/2}]^{-\alpha}\,[1+t+(1-2xt+t^2)^{1/2}]^{-\beta}$$

$$= \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n.$$

Let $f(x)$ be a function defined on the interval $-1 \leqslant x \leqslant 1$ such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha}\,(1+x)^{\beta} f(x)\, dx \qquad \ldots(1.1)$$

exists in the sense of Lebesgue. The Fourier-Jacobi series corresponding to the function $f(x)$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad \ldots(1.2)$$

where

$$a_n = \frac{1}{g_n}\int_{-1}^{1} (1-t)^{\alpha}\,(1+t)^{\beta} f(t)\, P_n^{(\alpha,\beta)}(t)\, dt$$

and

$$g_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\cdot\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\cdot\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}.$$

§2. Let $\Sigma\, b_n$ be a given infiniteseries with the sequences of partial sums $\{S_n\}$. Let $\{p_n\}$ be sequence of constants real or complex and let us write,

$$P_n = \sum_{\nu=0}^{n} p_\nu.$$

The sequence to sequence transformation

$$t_n = \sum_{\nu=0}^{n} \frac{p_{n-\nu} S_\nu}{P_n} = \sum_{\nu=0}^{n} \frac{p_\nu S_{n-\nu}}{P_n},\ (P_n \neq 0) \qquad \ldots(2.1)$$

defines the sequence $\{t_n\}$ of Nörlund means (Nörlund 1919) of the sequence $\{S_n\}$ generated by the sequence of coefficient $\{p_n\}$. The sesies $\Sigma\, b_n$ is said to be summable (N, p_n) to sum S if $\lim_{n\to\infty} t_n$ exists and equals S (Gupta 1970, p. 64).

The conditions for regularity of the method of summability (N, p_n) defined by (2.1) are

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \qquad \ldots(2.2)$$

and

$$\sum_{k=0}^{n} |\, p_k \,| = O(\,|\, P_n \,|\,) \text{ as } n \to \infty. \qquad \ldots(2.3)$$

When $p_n = \frac{1}{n+1}$, (N, p_n) summability reduces to Harmonic summability (H) (Gupta 1970, p. 110).

§3. Recently, Sharma (1976) has established a theorem on the Nörlund summability of Fourier-Jacobi series. This result is analogous of the result of Hsiang (1969) for trigonometric-Fourier series.

Looking into the proof of the theorem of Sharma, we observe that he has used an additional condition

$$\sum_{k=a}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \text{ as } n \to \infty$$

without mentioning it in the statement of his theorem.

The object of our paper is to extend the theorem of Sharma. Our theorem includes the theorem of Sharma as particular case when $\theta(P_\tau) = P_\tau$, $\tau = [1/t]$ where $[\lambda]$ denotes the integral part of λ.

Now we establish the following theorem.

Theorem — If

$$F_1(t) = \int_0^t |\, F(\phi) \,|\, d\phi = O\left(\frac{\psi(\tau)\, t^{2\alpha+2}}{\theta(P_\tau)}\right) \text{ as } t \to 0 \qquad \ldots(3.1)$$

where

$$F(\phi) = [f(\cos\phi) - A] (\sin \tfrac{1}{2}\phi)^{2\alpha+1} (\cos \tfrac{1}{2}\phi)^{2\beta+1}$$

and $\psi(t)$ and $\theta(t)$ are non-negative monotonic increasing functions of t such that

$$\psi(n) \log n = O(\theta(P_n)) \text{ as } n \to \infty \quad \text{...(3.2)}$$

$$n^{(2\alpha+1)/2} = o(P_n) \text{ as } n \to \infty \quad \text{...(3.3)}$$

and

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \text{ as } n \to \infty \quad \text{...(3.4)}$$

then the series (1.2) is summable (N, p_n) at the point $x = +1$, to sum A, provided that the condition $-\frac{1}{2} \leqslant \alpha < \frac{1}{2}$, $\beta > -\frac{1}{2}$ and the antipole condition

$$\int_{-1}^{b} (1 + x)^{(2\beta-3)/4} \mid f(x) \mid dx < \infty \quad \text{...(3.5)}$$

are satisfied, where b is fixed and (N, p_n) is regular Nörlund method defined by the real non-negative and non-increasing sequence $\{p_n\}$ such that $p_n \to \infty$ as $n \to \infty$.

Note: It may be noted here that anologous theorem to Theorem 1 can be stated for the other end point -1 by suitable adjustment in α and β.

§4. The following lemmas are necessary to prove our theorem.

Lemma 1 (Gupta 1970, p. 79) — Let

$$N_n(\phi) = \frac{2^{2+\beta}}{P_n} \sum_{k=0}^{n} p_k \lambda_{n-k} P_{n-k}^{(\alpha+1,\beta)} (\cos\phi)$$

where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} = \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} \cdot n^{\alpha+1}.$$

Then

(i) for $0 \leqslant \phi \leqslant \dfrac{1}{n}$,

$$\mid N_n(\phi) \mid = O(n^{2\alpha+2}) \quad \text{...(4.1)}$$

(ii) for $\dfrac{1}{n} \leqslant \phi \leqslant \pi - \dfrac{1}{n}$, $\alpha \geqslant -\frac{1}{2}$,

$$\mid N_n(\phi) \mid = \frac{1}{P_n} O\left(\frac{n^{(2\alpha+1)/2} P_{[1/\phi]}}{(\sin\frac{1}{2}\phi)^{(2\alpha+3)/2} (\cos\frac{1}{2}\phi)^{(2\beta+1)/2}}\right) + O\left(\frac{n^{(2\alpha-1)/2}}{(\sin\frac{1}{2}\phi)^{(2\alpha+5)/2} (\cos\frac{1}{2}\phi)^{(2\beta+3)/2}}\right) \quad \text{...(4.2)}$$

(iii) for $\pi - \frac{1}{n} \leqslant \phi \leqslant \pi,\ \alpha \geqslant -\frac{1}{2},\ \beta > -\frac{1}{2}$

$$| N_n(\phi) | = O(n^{\alpha+\beta+1}). \qquad ...(4.3)$$

Lemma 2 (Gupta 1970, p. 81) — The antipole condition

$$\int_{-1}^{b} (1 + x)^{(2\beta-3)/4} | f(x) | \, dx < \infty$$

means

$$\int_{a=\cos^{-1} b}^{\pi} (\cos \tfrac{1}{2} t)^{(2\beta-1)/2} | f(\cos t) - A | \, dt < \infty \qquad ...(4.4)$$

which is further

$$\int_{0}^{1/n} t^{(2\beta-1)/2} | f(- \cos t) - A | \, dt = o(1) \qquad ...(4.5)$$

as $n \to \infty$.

§5. *Proof of the Theorem* — Following the lines of Obrechkoff (1936, p. 99) the nth partial sum of the series (1.2) at the point $x = +1$ is given by

$$S_n(1) = 2^{\alpha+\beta} \int_{0}^{\pi} (\sin \tfrac{1}{2} \phi)^{2\alpha} (\cos \tfrac{1}{2} \phi)^{2\beta} f(\cos \phi) \, S_n(1, \cos \phi) \sin \phi \, d\phi \qquad ...(5.1)$$

where $S_n(1, \cos \phi)$ denotes the nth partial sum of the series

$$\sum_{m} \frac{P_m^{(\alpha,\beta)}(1) \, P_m^{(\alpha,\beta)}(\cos \phi)}{g_m}$$

where

$$g_m = \frac{2^{\alpha+\beta+1}\Gamma(m + \alpha + 1) \, \Gamma(m + \beta + 1)}{(2m + \alpha + \beta + 1) \, \Gamma(m + 1) \, \Gamma(m + \alpha + \beta + 1)}.$$

Rao (1929) has shown that

$$S_n(1, \cos \phi) = \lambda_n P_n^{(\alpha+1,\beta)}(\cos \phi)$$

where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}\Gamma(n + \alpha + \beta + 2)}{\Gamma(\alpha + 1) \, \Gamma(n + \beta + 1)} = \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha + 1)} \cdot n^{\alpha+1}.$$

Therefore

$$S_n(1) - A = 2^{\alpha+\beta+1}\lambda_n \int_0^{\pi} (\sin \tfrac{1}{2}\phi)^{2\alpha+1} (\cos \tfrac{1}{2}\phi)^{2\beta+1} [f(\cos\phi) - A] \times P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi \qquad \ldots(5.2)$$

$$= 2^{\alpha+\beta+1} \lambda_n \int_0^{\pi} F(\phi)\, P_n^{(\alpha+1,\beta)}(\cos\phi)\, d\phi.$$

The Nörlund mean of the series (1.2) at the point $x = +1$ is

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k S_{n-k}(1)$$

or

$$t_n - A = \frac{1}{P_n} \sum_{k=0}^{n} p_k (S_{n-k}(1) - A)$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_k \,.\, 2^{\alpha+\beta+1}\lambda_{n-k} \int_0^{\pi} F(\phi)\, P_{n-k}^{(\alpha+1,\beta)}(\cos\phi)\, d\phi$$

$$= \int_0^{\pi} F(\phi)\, N_n(\phi)\, d\phi. \qquad \ldots(5.3)$$

To prove our theorem we have to show that

$$I = \int_0^{\pi} F(\phi)\, N_n(\phi)\, d\phi = o(1) \text{ as } n \to \infty.$$

We write,

$$I = \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-(1/n)} + \int_{\pi-(1/n)}^{\delta}\right) F(\phi)\, N_n(\phi)\, d\phi$$

(where δ is an adjusted constant)

$$= I_1 + I_2 + I_3 + I_4 \text{ (say)}. \qquad \ldots(5.4)$$

Applying (4.1), we have

$$|I_1| = O\left(n^{2\alpha+2} o\left(\frac{\psi(n)}{\theta(P_n)}\right) . n^{-2\alpha-2}\right)$$

$$= o\left(\frac{\psi(n)}{\theta(P_n)}\right)$$

$$= o(1) \text{ as } n \to \infty \text{ by the hypothesis (3.2).} \qquad \ldots(5.5)$$

Again by the application of (4.2)

$$| I_2 | = O\left(\int_{1/n}^{\delta} | F(\phi) | \frac{n^{(2\alpha+1)/2}}{P_n} \cdot P_{[1/\phi]} (\sin \tfrac{1}{2}\phi)^{(-2\alpha-3)/2} \, d\phi \right)$$

$$+ O\left(\int_{1/n}^{\delta} | F(\phi) | \, n^{(2\alpha-1)/2} (\sin \tfrac{1}{2}\phi)^{(-2\alpha-5)/2} \, d\phi \right)$$

$$= I_{2\cdot1} + I_{2\cdot2} \text{ (say).} \qquad \ldots(5.6)$$

Now

$$| I_{2\cdot1} | = O\left(\frac{n^{(2\alpha+1)/2}}{P_n} \int_{1/n}^{\delta} | F(\phi) | \frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}} \, d\phi \right)$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) \left[o\left(\frac{\psi([1/\phi]) \, \phi^{2\alpha+2}}{\theta(P_{[1/\phi]})} \right) \cdot \frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}} \right]_{1/n}^{\delta}$$

$$+ O\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) \int_{1/n}^{\delta} o\left(\frac{\psi([1/\phi]) \, \phi^{2\alpha+2}}{\theta(P_{[1/\phi]})} \right) \frac{d}{d\phi} \left(\frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}} \right) d\phi$$

$$= I_{2,1,1} + I_{2,1,2} \text{ (say).} \qquad \ldots(5.7)$$

But

$$I_{2,1,1} = O\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) \left[o\left(\frac{\psi([1/\phi]) \, \phi^{(2\alpha+1)/2}}{\theta(P_{[1/\phi]}} P_{[1/\phi]} \right) \right]_{1/n}^{\delta}$$

$$= o\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) + o\left(\frac{\psi(n)}{\theta(P_n)} \right)$$

$$= o(1) \text{ as } n \to \infty. \qquad \ldots(5.8)$$

Now

$$I_{2,1,2} = O\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) \int_{1/n}^{\delta} o\left(\frac{\psi([1/\phi]) \, \phi^{2\alpha+2}}{\theta(P_{[1/\phi]})} \right) \frac{d}{d\phi} \left(\frac{P_{[1/\phi]}}{\phi^{(2\alpha+3)/2}} \right) d\phi$$

$$= o\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) \int_{1/\delta}^{n} \frac{\psi([x])}{\theta(P_{[x]})} \, x^{-2\alpha-2} \frac{d}{dx} (P_{[x]} x^{(2\alpha+3)/2}) \, dx$$

$$= o\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) + o\left(\frac{n^{(2\alpha+1)/2}}{P_n} \right) \sum_{k=a}^{n} \frac{\psi(K)}{\theta(P_k)} k^{-2\alpha-2} \triangle (P_k k^{(2\alpha+3)/2})$$

where $a = [\delta^{-1}] + 1$ and $\triangle P_k = P_{k+1} - P_k$

$$= o\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) + o\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \sum_{k=a}^{n} \frac{P_k}{\log k} \cdot \frac{1}{k^{(2\alpha+1)/2}}$$

$$= o\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) + o\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \text{ by (3.4)}$$

$$= o(1) \text{ as } n \to \infty. \qquad \text{...(5.9)}$$

Now we consider $I_{2,2}$ where

$$I_{2,2} = O\left(\int_{1/n}^{\delta} | F(\phi) | \, n^{(2\alpha-1)/2} (\sin \tfrac{1}{2}\phi)^{(-2\alpha-5)/2} \, d\phi\right)$$

$$= o(n^{(2\alpha-1)/2}) \left[\frac{\psi([1/\phi]) \cdot \phi^{(2\alpha-1)/2}}{\theta(P_{[1/\phi]})}\right]_{1/n}^{\delta}$$

$$+ o(n^{(2\alpha-1)/2}) \int_{1/n}^{\delta} \frac{\psi([1/\phi])}{\theta(P_{[1/\phi]})} \phi^{(2\alpha-3)/2} \, d\phi$$

$$= o(n^{(2\alpha-1)/2}) + o\left(\frac{\psi(n)}{\theta(P_n)}\right) + o(n^{(2\alpha-1)/2} \int_{1/\delta}^{n} \frac{\psi([x])}{\theta(P_{[x]})} x^{(-2\alpha-1)/2} \, dx$$

[Integrating by parts and applying (3.1)]

$$= o(1) \text{ as } n \to \infty \text{ because } \alpha < \tfrac{1}{2}. \qquad \text{...(5.10)}$$

Now we consider I_3, where

$$| I_3 | = O\left(\int_{\delta}^{\pi-(1/n)} | F(\phi) | \frac{n^{(2\alpha+1)/2}}{P_n} \cdot P_{[1/\phi]} \frac{d\phi}{(\sin \frac{1}{2}\phi)^{(2\alpha+3)/2} (\cos \frac{1}{2}\phi)^{(2\beta+1)/2}}\right.$$

$$+ O(n^{(2\alpha-1)/2} \int_{\delta}^{\pi-(1/n)} | F(\phi) | \frac{d\phi}{(\sin \frac{1}{2}\phi)^{(2\alpha+5)/2} (\cos \frac{1}{2}\phi)^{(2\beta+3)/2}}$$

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) \int_{\delta}^{\pi-(1/n)} | f(\cos\phi) - A | \, (\cos \tfrac{1}{2}\phi)^{(2\beta-1)/2} \cdot (\cos \tfrac{1}{2}\phi) \, d\phi$$

$$+ O(n^{(2\alpha-1)/2}) \int_{\delta}^{\pi-(1/n)} | f(\cos\phi) - A | \, (\cos \tfrac{1}{2}\phi)^{(2\beta-1)/2} \, d\phi$$

(*equation continued on p.* 1310)

$$= O\left(\frac{n^{(2\alpha+1)/2}}{P_n}\right) + O(n^{(2\alpha-1)/2}) \quad \text{From (4.4)}$$

$$= o(1) \text{ as } n \to \infty,\ \alpha < \tfrac{1}{2}. \qquad \ldots(5.11)$$

Lastly

$$| I_4 | = \left| \int_{\pi-(1/n)}^{\pi} F(\phi)\, N_n(\phi)\, d\phi \right|$$

$$= O(n^{\alpha+\beta+1}) \int_{\pi-(1/n)}^{\pi} | f(\cos\phi) - A |\, (\cos \tfrac{1}{2}\phi)^{2\beta+1} \,.\, (\sin \tfrac{1}{2}\phi)^{2\alpha+1}\, d\phi$$

$$= O(n^{(2\alpha-1)/2}) \int_{0}^{1/n} | f(-\cos\phi) - A |\, \phi^{(2\beta-1)/2}\, d\phi.$$

$$= o(1) \text{ by the application of (4.5).} \qquad \ldots(5.12)$$

Combining (5.4) to (5.12)

$$I = o(1) \text{ as } n \to \infty.$$

This completes the proof of the theorem.

Example 1 — If we take

$$f(x) = A + 2(\cos^{-1} x)^2, \quad x \in [-1, 1]$$

$$\psi(t) = \frac{t}{\log(2+t)},\ \theta(t) = t,\ t \in [0, \infty)$$

and $p_n = 1$

then all the conditions given in our theorem are satisfied.

Example 2 — If we take

$$f(x) = A + B(\cos^{-1} x)^r,\ r > 0,\ x \in [-1, 1]$$

where A and B are constants, and

$$\psi(t) = \frac{t^\lambda}{\log(2+t)},\ \theta(t) = t^\lambda,\ t \in [0, \infty)$$

and p_n is such that $P_n = n^p$, $p > 1$

then all the conditions given in our theorem are satisfied, if λ is such that $p\lambda - \lambda \geqslant 0$.

From above various examples can be constructed by giving suitable values to r, λ and p.

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REFERENCES

Gupta, D. P. (1970). D.Sc. Thesis, Allahabad University.

Hardy, G. H. (1949). Divergent series. The University Press, Oxford.

Hsiang, F. C. (1969). On Nörlund summability of Fourier series. *Bull. Calcutta math. Soc.*, **61** (1), 1-5.

Nörlund, N. E. (1919). Sur Une application des function Permutables. *Lunda Univ. Arsekrifts* (2), **16**, No. 3.

Obrechkoff, N. (1936). Formules asymototiques pour les polynomes de Jacobi et sur les series suivant les memes polynomes (Russian). *Ann. Univ. Sofia Fac. Phy. Math.*, **32**, 39-135.

Rao, H. (1929). Über die Lebesgueschen Konstanten der Reihenenwick tungen nach Jacobischen polynomen. *J. Reine. angew. Math.*, **161**, 237-54.

Sharma, M. M. (1976). On the Nörlund summability of Fourier-Jacobi series. *Vijnana Parishad Anusandhan Patrika*, **2** (19), 143-52.