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INTEGRABILITY OF TRIGONOMETRIC SERIES

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This note deals with the integrability of trigonometric series with δ-quasi-monotone coefficients. We have generalized the theorems of Yong (1965) in two directions.

§1. A sequence $\{a_n\}$ is said to be quasi-monotone if $n^{-\beta} a_n$ is monotonically decreasing to zero for some $\beta > 0$, or equivalently, if $a_n \geqslant 0$ and $a_{n+1} \leqslant a_n(1 + (\alpha/n))$ for some constant $\alpha > 0$ for all $n \geqslant n_0(\alpha)$ (Shah 1947, Szász 1948). It is clear that every monotone decreasing sequence is quasi-monotone and the converse is not necessarily true. Quasi-monotone sequences are known to share many of the properties of decreasing sequences, for example, Olivier's theorem (Szász 1948) the Cauchy condensation test for convergence (Shah 1947, Szász 1948); and a number of theorems about trigonometric series (Shah 1962, Yong 1965, 1966).

The concept of quasi-monotonicity was further generalized by Boas (1965) in the following manner: 'A sequence $\{a_n\}$ will be called δ-quasi-monotone if $a_n \to 0$ as $n \to \infty$, $a_n > 0$ ultimately, $\triangle a_n = a_n - a_{n+1} \geqslant -\delta_n$ for some positive sequence $\{\delta_n\}$. It is clear that a quasi-monotone sequence with $a_n \to 0$ is a δ-quasi-monotone sequence when $\delta_n = \alpha a_n/n$.'

By $\phi(x) \sim [a, b]$, $0 \leqslant a \leqslant b < \infty$ or $-\infty < a \leqslant b \leqslant 0$, we denote a non-negative function $\phi(x)$, not identically zero, such that $x^{-a}\, \phi(x)$ is non-decreasing and $x^{-b}\, \phi(x)$ is non-increasing as x increases in $(0, \infty)$. By $\phi(x) \sim \langle\, a, b \,\rangle$, we denote a function $\phi(x)$ such that for some positive ϵ, $\phi(x) \sim [a + \epsilon, b - \epsilon]$ (Chen 1965).

A positive, continuous function $L(x)$ defined for $x > 0$ is said to be slowly varying if

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1$$

for every fixed $t > 0$ (Karamata 1933).

§2. Concerning the integrability of trigonometric series, Yong (1965) proved the following theorems.

Theorem A — Let $0 < r < 1$, and $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$ as $n \to \infty$. Then $\sum_1^\infty n^{r-1}L(n)\, a_n$ converges, if and only if $\frac{1}{2} a_0 + \sum_1^\infty a_n \cos nx$ converges everywhere to $f(x)$ save possibly at $x = 0$, and

$$x^{-r}L(1/x)\, f(x) \in L(0, \pi).$$

Theorem B — Let $\{a_n\}$ be a quasi-monotone sequence with $a_n \to 0$ as $n \to \infty$.

(i) For $0 < r < 2$, if $\sum_1^\infty n^{r-1}L(n)\, a_n$ converges, then $\sum_1^\infty a_n \sin nx$ converges everywhere to $g(x)$, and

$$x^{-r}L(1/x)\, g(x) \in L(0, \pi).$$

(ii) For $0 < r < 1$, if $\sum_1^\infty a_n \sin nx$ converges everywhere to $g(x)$, and

$$x^{-r}L(1/x)\, g(x) \in L(0, \pi), \text{ then } \sum_1^\infty n^{r-1}L(n)\, a_n$$

converges.

§3. The object of this paper is to generalize these theorems. In what follows we shall establish the following theorems.

Theorem 1 — Let $\phi(x) \sim \langle -1, 0 \rangle$. Let $\{a_n\}$ be a δ-quasi-monotone sequence and $\sum_{n=1}^\infty \phi(1/n)\, \delta_n < \infty$. Then $\sum_{n=1}^\infty (1/n)\, \phi(1/n)\, a_n$ is convergent if and only if $\frac{1}{2} a_0 + \sum_{n=1}^\infty a_n \cos nx$ converges to $f(x)$ except possibly at $x = 0$, and

$$\phi(x)\, f(x) \in L(0, \pi).$$

Theorem 2 — (i) Let $\phi(x) \sim \langle -2, -1 \rangle$. Let $\{a_n\}$ be a δ-quasi-monotone sequence. If $\sum_{n=1}^\infty \phi(1/n)\, \delta_n$ and $\sum_{n=1}^\infty (1/n)\, \phi(1/n)\, a_n$ are convergent, then $\sum_{n=1}^\infty a_n \sin nx$ converges everywhere to $g(x)$ and $\phi(x)\, g(x) \in L(0, \pi)$.

(ii) Let $\phi(x) \sim \langle -1, 0 \rangle$. If $\sum_{n=1}^\infty a_n \sin nx$ converges everywhere to $g(x)$ and $\phi(x)\, g(x) \in L(0, \pi)$, then $\sum_{n=1}^\infty (1/n)\, \phi(1/n)\, a_n$ converges.

As a particular case in Theorem 1 and Theorem 2(ii), we may set $\phi(x) = x^{-r}L(1/x)$, where $0 < r < 1$ and in Theorem 2(i) we may set $\phi(x) = x^{-r}L(1/x)$, where $0 < r < 2$.

§4. For the proof of these theorems we require the following lemmas.

Lemma 1 — Let $\phi(x) \sim \langle -1, 0 \rangle$ or $\phi(x) \sim \langle -2, -1 \rangle$, then

$$K_1\phi(1/n) \leqslant \sum_{k=1}^{n} (1/k)\, \phi(1/k) \leqslant K_2\phi(1/n)$$

where K_1 and K_2 are two positive constants.

Proof : Let $\phi(x) \sim \langle -1, 0 \rangle$, then

$$\begin{aligned}
\sum_{k=1}^{n} (1/k)\, \phi(1/k) &= \sum_{k=1}^{n} \frac{k^{\epsilon}}{k^{\epsilon}} \cdot (1/k)\, \phi(1/k) \\
&= \sum_{k=1}^{n} \frac{1}{k^{1-\epsilon}}\, \phi(1/k)\, k^{-\epsilon} \\
&\geqslant \frac{1}{n^{1-\epsilon}}\, \phi(1/n) \sum_{k=1}^{n} k^{-\epsilon} \\
&\geqslant K_1 \frac{1}{n^{1-\epsilon}}\, \phi(1/n)\, n^{1-\epsilon} \\
&= K_1\phi(1/n).
\end{aligned}$$

Again

$$\begin{aligned}
\sum_{k=1}^{n} (1/k)\, \phi(1/k) &= \sum_{k=1}^{n} \frac{k^{\epsilon}}{k^{\epsilon}} \cdot (1/k)\, \phi(1/k) \\
&= \sum_{k=1}^{n} \frac{1}{k^{\epsilon}}\, \phi(1/k)\, k^{\epsilon-1} \\
&\leqslant \frac{1}{n^{\epsilon}}\, \phi(1/n) \sum_{k=1}^{n} k^{\epsilon-1} \\
&\leqslant K_2 \frac{1}{n^{\epsilon}}\, \phi(1/n)\, n^{\epsilon} \\
&= K_2\phi(1/n).
\end{aligned}$$

Thus the result follows. Similarly we can prove the result for $\phi(x) \sim \langle -2, -1 \rangle$.

Lemma 2 — Let $\{a_n\}$ be a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} \phi(1/n)\, \delta_n < \infty$. If $\sum_{n=1}^{\infty} (1/n)\, \phi(1/n)\, a_n$ converges, then $\sum_{n=1}^{\infty} \phi(1/n) \mid \triangle a_n \mid$ converges, where

$$\phi(x) \sim \langle -1, 0 \rangle \quad \text{or} \quad \phi(x) \sim \langle -2, -1 \rangle.$$

PROOF: Since $\triangle a_n \geqslant -\delta_n$, we have $\mid \triangle a_n \mid \leqslant \triangle a_n + 2\delta_n$. Suppose that $\sum_{n=1}^{\infty} (1/n)\, \phi(1/n)\, a_n < \infty$ then by Lemma 1,

$$\begin{aligned}
\sum_{n=1}^{\infty} \phi(1/n) \mid \triangle a_n \mid &\leqslant K \sum_{n=1}^{\infty} \mid \triangle a_n \mid \sum_{j=1}^{\infty} (1/j)\, \phi(1/j) \\
&\leqslant K \sum_{n=1}^{\infty} (\triangle a_n + 2\delta_n) \sum_{j=1}^{n} (1/j)\, \phi(1/j) \\
&= K \sum_{j=1}^{\infty} (1/j)\, \phi(1/j) \sum_{n=j}^{\infty} (\triangle a_n + 2\delta_n) \\
&= K \sum_{j=1}^{\infty} (1/j)\, \phi(1/j)\, (a_j + 2 \sum_{n=j}^{\infty} \delta_n) \\
&\leqslant K \sum_{j=1}^{\infty} (1/j)\, \phi(1/j)\, a_j + 2K \sum_{n=1}^{\infty} \delta_n \sum_{j=1}^{n} (1/j)\, \phi(1/j) \\
&\leqslant K \sum_{j=1}^{\infty} (1/j)\, \phi(1/j)\, a_j + K \sum_{n=1}^{\infty} \phi(1/n)\, \delta_n \\
&< \infty.
\end{aligned}$$

Lemma 3 (Chen 1965) — Let $\psi(x) = (1/x)\, \phi(1/x)$, where $\phi(x) \sim \langle -1, 0 \rangle$. Then $\psi(x) \sim \langle -1, 0 \rangle$.

Lemma 4 (Chen 1965) — If $\psi(x) \sim \langle -1, 0 \rangle$ and $(1/x)\, \psi(1/x) f(x) \in L(0, \pi)$ and if $a_n = (2/\pi) \int_0^{\pi} f(x) \cos nx \, dx$ for every n, then $\sum_{n=1}^{\infty} \psi(n)\, a_n$ is convergent.

Lemma 5 (Chen 1965) — If $a_n \geqslant 0$ and if the series $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to the function $f(x)$ such that $\phi(x)\, f(x) \in L(0, \pi)$, where $\phi(x) \sim \langle -1, 0 \rangle$, then $\sum_{n=1}^{\infty} (1/n)\, \phi(1/n)\, a_n < \infty$.

Proof of Theorem 1 — Since $\triangle a_n \geqslant -\delta_n$, we have $|\triangle a_n| \leqslant \triangle a_n + 2\delta_n$. Also the convergence of the series $\sum_{n=1}^{\infty} \phi(1/n)\,\delta_n$ implies that $\sum_{n=1}^{\infty} \delta_n < \infty$. Therefore, using the condition that $a_n \to 0$, we have

$$\sum_{n=1}^{\infty} |\triangle a_n| \leqslant \sum_{n=1}^{\infty} \triangle a_n + 2 \sum_{n=1}^{\infty} \delta_n < \infty$$

and thus $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges to $f(x)$ except possibly at $x = 0$. Now

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \tfrac{1}{2} a_0 + \frac{1}{\sin \frac{1}{2} x} \sum_{n=1}^{\infty} \triangle a_n \cos \frac{(n+\frac{1}{2})\,x}{2} \sin \tfrac{1}{2} nx$$

so that

$$|f(x)| \leqslant \tfrac{1}{2} a_0 + (\pi/x) \sum_{n=1}^{\infty} |\triangle a_n| \, |\sin \tfrac{1}{2} nx| .$$

Thus

$$\int_0^{\pi} \phi(x)\, |f(x)|\, dx$$

$$\leqslant \tfrac{1}{2} a_0 \int_0^{\pi} \phi(x)\, dx + \pi \int_0^{\pi} x^{-1}\phi(x) \sum_{n=1}^{\infty} |\triangle a_n| \, |\sin \tfrac{1}{2} nx|\, dx$$

$$= \tfrac{1}{2} a_0 \int_0^{\pi} \phi(x)\, dx + \pi \sum_{n=1}^{\infty} |\triangle a_n| \int_0^{\pi} x^{-1}\phi(x)\, |\sin \tfrac{1}{2} nx|\, dx$$

$$= I_1 + I_2, \text{ say.}$$

Now

$$I_1 = \tfrac{1}{2} a_0 \int_0^{\pi} \phi(x)\, dx$$

$$= \tfrac{1}{2} a_0 \int_0^{\pi} x^{1-\epsilon}\phi(x)\, x^{\epsilon-1}\, dx$$

$$\leqslant \tfrac{1}{2} a_0 \pi^{1-\epsilon}\phi(\pi) \int_0^{\pi} x^{\epsilon-1}\, dx$$

$$= \frac{\pi}{2\epsilon}\, a_0 \phi(\pi)$$

and

$$I_2 = \pi \sum_{n=1}^{\infty} | \triangle a_n | \int_0^{\pi} x^{-1}\phi(x) | \sin \tfrac{1}{2} nx | \, dx$$

$$= \pi \sum_{n=1}^{\infty} | \triangle a_n | [\int_0^{1/n} x^{-1}\phi(x) | \sin \tfrac{1}{2} nx | \, dx + \int_{1/n}^{\pi} x^{-1}\phi(x) \times | \sin \tfrac{1}{2} nx | \, dx]$$

$$\leqslant \pi \sum_{n=1}^{\infty} | \triangle a_n | [(n/2) \int_0^{1/n} \phi(x) \, dx + \int_{1/n}^{\pi} x^{-1}\phi(x) \, dx]$$

$$= \frac{3\pi}{2\epsilon} \sum_{n=1}^{\infty} \phi(1/n) | \triangle a_n | .$$

Hence

$$\int_0^{\pi} \phi(x) | f(x) | \, dx \leqslant \frac{\pi}{2\epsilon} a_0 \phi(\pi) + \frac{3\pi}{2\epsilon} \sum_{n=1}^{\infty} \phi(1/n) | \triangle a_n |$$

$$< \infty \text{ by virtue of Lemma 2.}$$

Conversely, suppose that $\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ converges everywhere to $f(x)$ except possibly at $x = 0$. Let $\phi(x) f(x) \in L(0, \pi)$, then $f(x) \in L(0, \pi)$ and therefore a_n are the Fourier cosine coefficients of $f(x)$, that is to say,

$$a_n = (2/\pi) \int_0^{\pi} f(x) \cos nx \, dx \; (n = 0, 1, 2, \ldots).$$

Then by Lemma 4, $\sum_{n=1}^{\infty} (1/n) \phi(1/n) a_n$ is convergent.

This completes the proof of Theorem 1.

Proof of Theorem 2(i) — Since $\triangle a_n \geqslant - \delta_n$, we have $| \triangle a_n | \leqslant \triangle a_n + 2\delta_n$. The convergence of $\sum_{n=1}^{\infty} \phi(1/n) \delta_n$ implies that $\sum_{n=1}^{\infty} \delta_n < \infty$. Therefore by using the condition that $a_n \to 0$, we have $\sum_{n=1}^{\infty} | \triangle a_n | < \infty$. Thus $\sum_{1}^{\infty} a_n \sin nx$ converges to $g(x)$ for every x.

Now

$$g(x) = \frac{1}{2 \sin \frac{1}{2} x} \sum_{n=1}^{\infty} \triangle a_n (\cos \tfrac{1}{2} x - \cos (n + \tfrac{1}{2}) x)$$

$$= - \tfrac{1}{2} \tan (\tfrac{1}{4} x) \sum_{n=1}^{\infty} \triangle a_n + \frac{1}{2 \sin \frac{1}{2} x} \sum_{n=1}^{\infty} \triangle a_n (1 - \cos (n + \tfrac{1}{2}) x)$$

and hence

$$| g(x) | \leqslant \tfrac{1}{2} a_1 \tan (\tfrac{1}{4} x) + \tfrac{1}{2} \operatorname{cosec} \tfrac{1}{2} x \sum_{n=1}^{\infty} | \triangle a_n | [1 - \cos (n + \tfrac{1}{2}) x].$$

Thus

$$\int_0^\pi \phi(x) \, | g(x) | \, dx \leqslant \tfrac{1}{2} a_1 \int_0^\pi \phi(x) \tan (\tfrac{1}{4} x) \, dx + \tfrac{1}{2} \sum_{n=1}^{\infty} | \triangle a_n |$$

$$\times \int_0^\pi \phi(x) \operatorname{cosec} \tfrac{1}{2} x \, [1 - \cos (n + \tfrac{1}{2}) x] \, dx$$

$$\leqslant \tfrac{1}{4} a_1 \int_0^\pi x\phi(x) \, dx + \tfrac{1}{2} \sum_{n=1}^{\infty} | \triangle a_n | \, [\int_0^{1/n} \phi(x) \operatorname{cosec} \tfrac{1}{2} x$$

$$\times \{1 - \cos (n + \tfrac{1}{2}) x\} \, dx + \int_{1/n}^\pi \phi(x) \operatorname{cosec} \tfrac{1}{2} x$$

$$\times \{1 - \cos (n + \tfrac{1}{2}) x\} \, dx].$$

Now

$$\int_0^\pi x\phi(x) \, dx = \int_0^\pi x^{2-\epsilon} \phi(x) \, x^{\epsilon-1} \, dx$$

$$\leqslant \pi^{2-\epsilon} \phi(\pi) \int_0^\pi x^{\epsilon-1} \, dx$$

$$\leqslant K$$

$$\int_0^{1/n} \phi(x) \operatorname{cosec} \tfrac{1}{2} x \, [1 - \cos (n + \tfrac{1}{2}) x] \, dx$$

$$\leqslant Kn^2 \int_0^{1/n} x\phi(x) \, dx = Kn^2 \int_0^{1/n} x^{2-\epsilon} \phi(x) \, x^{\epsilon-1} \, dx$$

$$\leqslant Kn^2 (1/n)^{2-\epsilon} \, \phi(1/n) \int_0^{1/n} x^{\epsilon-1} \, dx = O(\phi(1/n)) \text{ as } n \to \infty$$

and

$$\int_{1/n}^{\pi} \phi(x) \operatorname{cosec} \tfrac{1}{2} x \, [1 - \cos (n + \tfrac{1}{2}) x] \, dx$$

$$\leqslant K \int_{1/n}^{\pi} x^{-1} \phi(x) \, dx = K \int_{1/n}^{\pi} x^{1+\epsilon} \phi(x) \, x^{-2-\epsilon} \, dx$$

$$\leqslant K(1/n)^{1+\epsilon} \, \phi(1/n) \int_{1/n}^{\pi} x^{-2-\epsilon} \, dx = O(\phi(1/n)) \quad \text{as} \quad n \to \infty.$$

Hence

$$\int_{0}^{\pi} \phi(x) \mid g(x) \mid dx < \infty, \text{ by virtue of Lemma 2.}$$

(ii) Suppose $\sum_{n=1}^{\infty} a_n \sin nx$ converges everywhere to $g(x)$ and let $\phi(x) g(x) \in L(0, \pi)$, then by Lemma 5, $\sum_{n=1}^{\infty} (1/n) \, \phi(1/n) \, a_n$ is convergent.

This completes the proof of Theorem 2.

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