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ON ASYMPTOTIC COMMUTATIVITY AND COMMON FIXED POINT THEOREMS

S. K. SAMANTA

Department of Mathematics, University of Burdwan, Burdwan (West Bengal) 713104

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Let $\mathcal{F}$ denote a family of self-mappings over a Banach space X. Two members f and g of $\mathcal{F}$ are said to be asymptotically commutative if $\lim_{i\to\infty} (f^i g(x) - g f^i(x)) = 0$ for all x in X. $\mathcal{F}$ is called asymptotically commutative iff any two members of $\mathcal{F}$ are so. With this concept of asymptotic commutativity as introduced by us in this paper we have proved simultaneous fixed point theorems for $\mathcal{F}$. If members of $\mathcal{F}$ are all linear, then our theorem generalizes well-known Markov-Kakutani Theorem. In case of members of $\mathcal{F}$ being nonlinear our results rest on two settings; one over a compact convex subset, and other over a closed convex subset of X. In both the cases our results include those already known for a commuting family $\mathcal{F}$. Examples have been cited to show the extent of generalization of theory of fixed points over a non-commuting family $\mathcal{F}$.

§1. It is known that Markov and Kakutani (Dunford and Schwartz 1958) have proved common fixed point theorem for a family of continuous linear mappings. Later on Marr (1963) has dealt with nonlinear mappings and has established common fixed point theorem for a family of nonexpansive self-mappings (i.e. mappings T for which $\| T(x) - T(y) \| \leqslant \| x - y \|, \forall\, x, y$) over a compact convex subset of a Banach space. In a noncompact setting Belluce and Kirk (1966) have proved common fixed point theorems for a finite family of nonexpansive mappings by introducing a concept known as 'normal structure' [for the definition of 'normal structure', see Kirk (1965)] in the space. Subsequently they have extended their results to arbitrary family of nonexpansive mappings by strengthening the hypothesis of normal structure to 'complete normal structure' [for the definition see Belluce and Kirk (1967)]. Also Kasahara (1966) has proved a common fixed point theorem for a family of nonexpansive mappings over a uniformly convex normed linear space. In all the above-mentioned works authors have worked always with a commuting family of mappings. Our aim in this paper is to establish common fixed point theorems for a noncommuting family of mappings. For doing so we have introduced a concept to be called as 'asymptotic commutativity' and with its aid have proved common fixed point theorems. In §2 we have established common fixed point theorems for a family of linear continuous mappings which gives Markov-Kakutani Theorem (Dunford and

Schwartz 1958) as a corollary. In §3 we have dealt with a family of nonlinear mappings and have proved simultaneous fixed point theorems which generalise those for commuting family of nonlinear mappings as referred to above. Examples have been given in support of theorems proved in this paper.

§2. We start with a definition.

Definition 2.1 — Let $(X, \mathcal{U})$ be a Hausdorff uniform space and $f, g : X \to X$ be two mappings. f and g are said to be pointwise asymptotically commutative if for each $x \in X$, the sequences $\{(f^i g(x), gf^i(x))\}$ and $\{(g^i f(x), fg^i(x))\}$ are eventually in every $u \in \mathcal{U}$.

We now prove the following theorem.

Theorem 2.1 — Let H be a nonempty compact convex subset of a Hausdorff linear topological space X, and $\mathcal{F}$ be a family of continuous and linear self-mappings over H, any two members of which are pointwise asymptotically commutative. Then $\mathcal{F}$ possesses a common fixed point.

PROOF : Using Zorn's Lemma, we get a nonempty compact convex subset K of H which is minimal with respect to being invariant under each member of $\mathcal{F}$. Take $x \in K$. Since K is compact and $T(K) \subseteq K$, let the sequence $\{T^{n_i}(x)\}$ be a convergent subsequence of $\{T^n(x)\}$ in K. We construct

$$K_1 = K(\{n_i\}, T) = \{z \in K \; ; \operatorname*{Lim}_i T^{n_i}(\xi) = z, \text{ for some } \xi \in K\}.$$

Clearly $K_1 \neq \phi$. Since T is linear and continuous, if follows that K_1 is convex. Next we show that K_1 remains invariant under each member of $\mathcal{F}$. Let $S \in \mathcal{F}$. If $z \in K_1$, then for some $\xi \in K$, $S(z) = S(\operatorname*{Lim}_i T^{n_i}(\xi) = \operatorname*{Lim}_i ST^{n_i}(\xi)$ (by continuity of S) $= \operatorname*{Lim}_i T^{n_i}S(\xi)$ (by asymptotic commutativity of S and T). Since $S(\xi) \in K$, we have $S(z) \in K(\{n_i\}, T) = K_1$. Since each member of $\mathcal{F}$ is continuous, it follows that $\overline{K(\{n_i\}, T)}$ (bar denoting the closure of a set) is a compact convex subset of K that remains invariant under each member of $\mathcal{F}$. By minimality of K we have $K = \overline{K(\{n_i\}, T)}$. Let f and g be two members of $\mathcal{F}$. By the argument given above there is a subsequence $\{m_i\}$ of $\{m\}$ such that $\overline{K(\{m_i\}, f)} = K$. Let $x \in K(\{m_i\}, f)$. Then $x = \operatorname*{Lim}_i f^{m_i}(\xi)$ for some $\xi \in K$. Using continuity and pointwise asymptotic cummutativity we have,

$$\begin{aligned} gf(x) &= gf(\operatorname*{Lim}_i f^{m_i}(\xi)) = g(\operatorname*{Lim}_i f^{m_i+1}(\xi)) = \operatorname*{Lim}_i gf^{m_i+1}(\xi) \\ &= \operatorname*{Lim}_i f^{m_i+1}g(\xi) = f(\operatorname*{Lim}_i f^{m_i}g(\xi)) = f(\operatorname*{Lim}_i gf^{m_i}(\xi)) \\ &= fg(\operatorname*{Lim}_i f^{m_i}(\xi)) = fg(x). \end{aligned}$$

This shows that f and g commute over $K(\{m_i\}, f)$ and hence f and g commute over $\overline{K(\{n_i\}, f)} = K$. Thus $\mathcal{F}$ is commutative over K and an application of Markov-Kakutani Theorem (Dunford and Schwartz 1958) completes the proof.

Corollary 2.1 [Markov-Kakutani Theorem (Dunford and Schwartz 1958)] — Let K be a nonempty compact convex subset of a Hausdorff linear topological space X and $\mathcal{F}$ be a commutative family of continuous linear self-mappings over K. Then $\mathcal{F}$ possesses a common fixed point.

§3. In this section we shall prove simultaneous fixed point theorem for non-linear mappings. Before we start, we give some definitions.

Definition 3.1 — Let (X, d) be a metric space and $f, g : X \to X$ be two mappings. f and g are said to be uniformly asymptotically commutative if $d(f^i g(x), gf^i(x)) \to 0$ and $d(g^i f(x), fg^i(x)) \to 0$ uniformly over X as $i \to \infty$.

Definition 3.2 — A mapping $f : X \to X$ is said to be asymptotically nonexpansive if for any $x, y \in X$, $d(f^i(x), f^i(y)) \leqslant K_i\, d(x, y)$ where $K_i \downarrow 1$ as $i \to \infty$.

Definition 3.3 — A bounded closed convex set K in a Banach space X is said to possess normal structure if for any closed convex subset H of K containing more than one point, there exists $x \in H$ such that $\sup_{y \in H} \| x - y \| < \delta(H)$, $\delta(H)$ denotes diameter of H. Before we prove Theorem 3.1 we need the following Lemma.

Lemma 3.1 — Let C be a nonempty closed subset of a complete metric space X and $f, g : C \to C$ be continuous mappings which are uniformly asymptotically commutative. If further

$$d(f^i g^i(x), (fg)^i(x)) \to 0 \text{ and } d(g^i f^i(x), (gf)^i(x)) \to 0 \qquad \ldots(A)$$

uniformly over C as $i \to \infty$, then

(i) $F(fg) = \{x \in C \,;\, fg(x) = x\} = F(gf) = \{x \in C \,;\, gf(x) = x\}$

(ii) $f : F(fg) \to F(fg)$ and $g : F(fg) \to F(fg)$

$$\left.\begin{array}{l}\text{(iii) } f^i g(x) = gf^i(x) \quad \text{and } g^i f(x) = fg^i(x) \\ \text{(iv) } f^i g^i(x) = (fg)^i(x) \text{ and } g^i f^i(x) = (gf)^i(x)\end{array}\right\}$$

for all $x \in F(fg)$.

PROOF : (i) Let $x \in F(fg)$. So, by (A), $f^i g^i(x) \to x$ uniformly. Hence $gf(f^i g^i(x)) \to gf(x)$ uniformly i.e. $gf^{i+1} g^i(x) \to gf(x)$ uniformly. Thus by asymptotic commutativity of f and g, we have $f^{i+1} g\, g^i(x) \to gf(x)$ uniformly. On the other-hand, $f^i g^i(x) \to x$ and hence $gf(x) = x = fg(x)$. This shows that $F(fg) \subseteq F(gf)$. Similarly $F(gf) \subseteq F(fg)$. This proves relation (i).

(ii) For $x \in F(fg)$, we have from (i), $fg(x) = gf(x) = x$; so,

$$gfg(x) = g^2 f(x) = g(x); \text{ i.e. } g(x) \in F(gf) = F(fg).$$

Similarly $f(x) \in F(fg)$. This proves (ii).

(iii) For $x \in F(fg)$, $g^i f(x) = g^{i-1} gf(x) = g^{i-1} fg(x) = g^{i-1}(x) = g^{i-2} g(x)$

$$= g^{i-2} fg(g(x)) = g^{i-2} fg^2(x).$$

Repeating the process we have

$$g^i f(x) = fg^i(x).$$

Similarly, $f^i g(x) = gf^i(x)$.

(iv) The relation $f^i g^i(x) = (fg)^i(x)$ is true for $i = 1$.

Suppose it is true for $i = i_0 - 1$, then

$$(fg)^{i_0}(x) = (fg)^{i_0-1}(fg)(x) = (fg)^{i_0-1}(x) = f^{i_0-1} g^{i_0-1}(x)$$

$$= f^{i_0-1} g^{i_0-1} gf(x) = f^{i_0-1} g^{i_0} f(x) = f^{i_0-1} f g^{i_0}(x) = f^{i_0} g^{i_0}(x).$$

This completes the induction. Similarly $(gf)^i(x) = g^i f^i(x)$.

Theorem 3.1 — Let K be a nonempty closed convex weakly compact subset of a Banach space X satisfying normal structure. Let $\mathcal{F} = (f_1, f_2, \ldots, f_n)$ be a finite family of nonexpansive mappings, any two members of which are uniformly asymptotically commutative, satisfying the following condition. For any sub-family $(f, g, \ldots, h)$ of $\mathcal{F}$,

$$(fg \ldots h)^i(x) - (f^i g^i \ldots h^i)(x) \to 0 \qquad \ldots \text{(B)}$$

uniformly over K.

Then $\mathcal{F}$ has a common fixed point.

PROOF : Using Zorn's Lemma, we find a minimal nonempty closed, convex subset M of K which remains invariant under each $f \in \mathcal{F}$. Let F denote the fixed point set of $(f_1 f_2 \ldots f_n)$. Clearly $F \neq \phi$ by Theorem 3 of Kirk (1965). For a cyclic permutation $\{f_{i+1}, f_{i+2}, \ldots, f_n, f_1, \ldots, f_i\}$ of $\{f_1, f_2, \ldots, f_n\}$ we show the followings:

$$f_{i+1}^j (f_{i+2} \ldots f_n f_1 \ldots f_i)(x) - (f_{i+2} \ldots f_n f_1 \ldots f_i) f_{i+1}^j (x) \to 0 \qquad \ldots (3.1)$$

uniformly over K as $j \to \infty$

and

$$f_{i+1}^j (f_{i+2} \ldots f_n f_1 \ldots f_i)^j (x) - (f_{i+1} f_{i+2} \ldots f_n f_1 \ldots f_i)^j (x) \to 0 \qquad \ldots (3.2)$$

uniformly over K as $j \to \infty$.

We establish (3.1) by method of finite induction. Suppose (3.1) is true for any m functions of $\mathcal{F}$, $1 \leqslant m < n$. Take $(m+1)$ number of functions in $\mathcal{F}$, and call them $f_1, f_2, \ldots, f_{m+1}$. Now for m functions we have

$$f^j_{i+1} \, (f_{i+3} \ldots f_{m+1} f_1 \ldots f_i)\,(x) - f_{i+3} \ldots f_{m+1} f_1 \ldots f_i \,(f^j_{i+1}\,(x)) \to 0 \qquad \ldots(3.\text{i})$$

uniformly over K as $j \to \infty$.

By asymptotic commutativity of f_{i+1} and f_{i+2}, we also have

$$f^j_{i+1}\, f_{i+2}\,(f_{i+3} \ldots f_{m+1} f_1 \ldots f_i(x)) - f_{i+2} f^j_{i+1}\,(f_{i+3} \ldots f_{m+1} f_1 \ldots f_i(x)) \to 0 \qquad \ldots(3.\text{ii})$$

uniformly over K as $j \to \infty$.

Since f_{i+2} is nonexpansive, we have from (3.i),

$$\| f_{i+2}\, f^j_{i+1}\,(f_{i+3} \ldots f_{m+1} f_1 \ldots f_i(x)) - f_{i+2} f_{i+3} \ldots f_{m+1} f_1 \ldots f_i f^j_{i+1}\,(x) \|$$

$$\leqslant \| f^j_{i+1}\,(f_{i+3} \ldots f_{m+1} f_1 \ldots f_i(x)) - f_{i+3} \ldots f_{m+1} f_1 \ldots f_i f^j_{i+1}\,(x) \| \to 0 \qquad \ldots(3.\text{iii})$$

uniformly as $j \to \infty$.

Hence (3.1) follows from (3.ii) and (3.iii).

To prove (3.2) we have by (B)

$$(f_{i+2} \ldots f_n f_1 \ldots f_i)^j\,(x) - (f^j_{i+2} \ldots f^j_n f^j_1 \ldots f^j_i)(x) \to 0 \qquad \ldots(3.\text{iv})$$

uniformly as $j \to \infty$

and

$$(f_{i+1} f_{i+2} \ldots f_n f_1 \ldots f_i)^j(x) - f^j_{i+1} f^j_{i+2} \ldots f^j_n f^j_1 \ldots f^j_i\,(x) \to 0 \qquad \ldots(3.\text{v})$$

uniformly as $j \to \infty$.

Since f^j_{i+1} is nonexpansive, we have from (3.iv)

$$f^j_{i+1}\,(f_{i+2} \ldots f_n f_1 \ldots f_i)^j\,(x) - f^j_{i+1} f^j_{i+2} \ldots f^j_n f^j_1 \ldots f^j_i\,(x) \to 0 \qquad \ldots(3.\text{vi})$$

uniformly as $j \to \infty$.

Thus from (3.v) and (3.vi) relation (3.2) is immediate.

Proceeding similarly as above we also have

$$(f_{i+2} \ldots f_n f_1 \ldots f_i)^j f_{i+1}\,(x) - f_{i+1}(f_{i+2} \ldots f_n f_1 \ldots f_i)^j\,(x) \to 0 \qquad \ldots(3.3)$$

uniformly over K as $j \to \infty$.

$$(f_{i+2} \dots f_n f_1 \dots f_i)^j f_{i+1}^j (x) - (f_{i+2} \dots f_n f_1 \dots f_i f_{i+1})^j (x) \to 0 \qquad \dots(3.4)$$

uniformly over K as $j \to \infty$.

Now by Lemma 2.1, we have for any $x \in F$,

$$x = f_1 f_2 \dots f_n (x) = f_2 f_3 \dots f_n f_1 (x) = \dots = f_{i+1} f_{i+2} \dots f_1 f_2 \dots f_i (x) \qquad \dots(3.\text{vii})$$

This shows that

$$f_i(x) = f_i(f_{i+1} \dots f_n f_1 \dots f_i(x)) = (f_i f_{i+1} \dots f_n f_1 \dots f_{i-1}) f_i(x)$$

i.e. $f_i(x)$ is a fixed point of $f_i f_{i+1} \dots f_n f_1 \dots f_{i-1}$ and therefore arguing similarly as above, $f_i(x)$ is a fixed point of $(f_1 f_2 \dots f_n)$. Thus $f_i(F) \subseteq F$, for $1 \leqslant i \leqslant m$. Further from relation (3.vii), it follows that $f_i(F) = F$, $1 \leqslant i \leqslant n$. The rest of the proof is a copy of arguments given by Belluce and Kirk (1966, Theorem 3) where by using normal structure hypothesis F has been shown to be a single-pointic set. Thus $\mathcal{F}$ has a common fixed point.

Corollary 3.1 (Theorem 3 of Belluce and Kirk 1966) — Suppose X is a nonempty weakly compact, convex subset of a Banach space B and suppose that X has normal structure. If $\mathcal{F}$ is a finite family of commuting nonexpansive mappings of X into itself then there is an $x \in X$ such that $f(x) = x$ for each $f \in \mathcal{F}$.

In case $\mathcal{F}$ contains an infinite number of members having properties as stated in Theorem 3.1 we have the following theorem whose proof is completed in two steps. Initially we apply Theorem 3.1 to obtain a fixed point for a finite sub-family of $\mathcal{F}$, and then we proceed exactly in the same manner as in the proof of Theorem of Belluce and Kirk (1967) to arrive at the desired conclusion.

Theorem 3.2 — Let K be a nonempty weakly compact convex subset of a Banach space X and suppose K has complete normal structure. Let $\mathcal{F}$ be an infinite family of nonexpansive self-mappings over K any two members of which are uniformly asymptotically commutative such that for any finite sub-family τ of $\mathcal{F}$, condition (B) of Theorem 3.1 is satisfied. Then $\mathcal{F}$ possesses a common fixed point.

However in a uniformly convex Banach space we have the following theorem.

Theorem 3.3 — Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $\mathcal{F}$ be a family of asymptotically nonexpansive mappings of K into itself such that any two members of $\mathcal{F}$ are pointwise asymptotically commutative, then $\mathcal{F}$ has a common fixed point.

PROOF : Let $f \in \mathcal{F}$, and F_f denote the set of fixed points of f. Since X is uniformly convex and f is asymptotically nonexpansive, F_f is nonempty bounded closed and convex (see Goebel and Kirk (1972), and therefore weakly compact. Now following the arguments used in the proof of Theorem 1 of Kasahara (1966) we show that $F = \bigcap_{f \in \mathcal{F}} F_f \neq \phi$. Thus F is the desired common fixed point set.

Over a compact setting of the domain space we have the following theorem.

Theorem 3.4 — Let X be a nonempty closed convex bounded subset of a Banach space B; let M be a compact subset of X. Let $\mathcal{F}$ be a nonempty family of nonexpansive mappings any two members of which are pointwise asymptotically commutative. Further if for some $f_1 \in \mathcal{F}$ and for each $x \in X$, $M \cap \overline{\left\{f_1^n(x)\right\}} \neq \phi$ then $\mathcal{F}$ has a common fixed point.

PROOF: Applying standard argument of Zorn's Lemma suppose X^* is a minimal nonempty closed convex subset of X which remains invariant under each $f \in \mathcal{F}$. Now f_1 satisfies all conditions of the Theorem of Kirk (1965) over X, and application of which shows that the fixed point set H of f_1 is a nonempty subset of $M^* = X^* \cap M$. Since any two members of $\mathcal{F}$ are pointwise asymptotically commutative $f(H) \subseteq H$ for each $f \in \mathcal{F}$. Let H^* be a subset of H which is minimal with respect to being nonempty closed and remaining invariant under each $f \in \mathcal{F}$. Take $f_0 \in \mathcal{F}$. Since H^* is compact $\{f_0^n(H^*)\}$ is a decreasing sequence of compact sets and hence $\bigcap_{n=1}^{\infty} \{f_0^n(H^*)\} \neq \phi$. Let $L_{f_0} = \bigcap_{n=1}^{\infty} \{f_0^n(H^*)\}$. We show that a point $x \in L_{f_0}$ iff there is $\xi = \xi(x)$ in H^* and a subsequence (depending on x) $\{f_0^{n_i}(\xi)\}$ of $\{f_0^n(\xi)\}$ such that $x = \operatorname{Lim}_i f_0^{n_i}(\xi)$. Let $x \in L_{f_0}$. Then for each n, there is $\xi_n \in H^*$ such that $x = f_0^n(\xi_n)$. Since H^* is compact, there is a convergent subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ converging to, say $\xi \in H^*$. Since f_0 is nonexpansive $\{f_0^{n_i}(\xi)\}$ converges to x. Conversely, let for some $\xi \in H^*$, $\{f_0^{n_i}(\xi)\}$ converges to x. Then for a fixed m, $f_0^{n_i}(\xi) \in f_0^m(H^*)$ for sufficiently large values of i. Since $f_0^m(H^*)$ is closed, we have $\operatorname{Lim}_i f_0^{n_i}(\xi) = x \in f_0^m(H^*)$. This being true for every m, we have $x \in L_{f_0}$. Next we show that L_{f_0} remains invariant under each $f \in \mathcal{F}$. Take $x \in L_{f_0}$. Suppose $x = \operatorname{Lim}_i f_0^{n_i}(\xi)$ for some $\xi \in H^*$. Since f is continuous, we have $f(x) = \operatorname{Lim}_i f f_0^{n_i}(\xi) = \operatorname{Lim} f_0^{n_i}(f(\xi))$ by pointwise asymptotic commutativity of f and f_0.

Since $f(\xi) \in H^*$ we have $f(x) \in L_{f_0}$. Thus $f(L_{f_0}) \subseteq L_{f_0}$ for all $f \in \mathcal{F}$. Further L_{f_0} is closed and therefore the minimality of H^* we have $L_{f_0} = H^*$. Hence $f_0(H^*) = H^*$. Since f_0 is arbitrary we have $f(H^*) = H^*$ for all $f \in \mathcal{F}$. The rest of the proof is similar to the proof as in Theorem 1 of Belluce and Kirk (1966).

Corollary 3.2 (Theorem 1 of Belluce and Kirk 1966) — Let X be a nonempty, bounded, closed, convex subset of a Banach space. Let M be a compact subset of X. Let $\mathcal{F}$ be a nonempty commutative family of nonexpansive self-mappings over K with the property that for some $f_1 \in \mathcal{F}$ and for each $x \in X$, the closure of the set $\{f^n(x) ; n = 1, 2, ...\}$ contains a point of M. Then there is a point $x \in M$ such that $f(x) = x$ for each $f \in \mathcal{F}$.

Corollary 3.3 (Theorem of Marr 1963) — Let B be a Banach space and let X be a nonempty compact subset of B. If $\mathcal{F}$ is a nonempty commutative family of nonexpansive mappings of X into itself, then the family $\mathcal{F}$ has a common fixed point in X.

Example 3.1 given below shows that there is a non-commuting family of mappings which is asymptotically commuting and that it illustrates situation in which Theorem 3.1 and Theorem 3.4 apply.

Example 3.1 — Let X be the space of reals with usual Euclidean norm and $K = [1, 2]$ be a subset of X. Define $f, g : K \to K$ as follows:

$$f(x) = 1 + \frac{x-1}{2}, \qquad \text{for all } x \in K$$

and

$$g(x) = 1 + \left(\frac{x-1}{2}\right)^2, \text{ for all } x \in K.$$

Then clearly $| f^i g(x) - g f^i(x) | = (x-1)^2 \left| \frac{1}{2^{i+2}} - \frac{1}{2^{2(i+2)}} \right| \to 0$

uniformly over K as $i \to \infty$, and

$$| f^i g^i(x) - (fg)^i(x) | = \frac{(x-1)^{2^i}}{2^{1+2+\cdots+2^{i-1}}} \times \left[\frac{1}{2^{i-1}\, 2^{2^i}} - \frac{1}{2^{2(1+2+\cdots+2^{i-1})}}\right] \to 0$$

uniformly over K as $i \to \infty$.

Similarly $| g^i f(x) - f g^i(x) | \to 0$ and $| g^i f^i(x) - (gf)^i(x) | \to 0$

uniformly over K as $i \to \infty$.

However, $$fg(x) = 1 + \frac{(x-1)^2}{2^3} \neq gf(x) = 1 + \frac{(x-1)^2}{2^4},$$

for all x. Further f and g are nonexpansive.

Finally Example 3.2 and Example 3.3 belows show how far the hypothesis of Theorems are independent.

Example 3.2 — Let $X = \{(r, \theta); 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \pi/2\}$. Then X is closed convex subset of E^2 with the usual Euclidean norm $\| (x, y) \| = \sqrt{x^2 + y^2}$, $(x, y) \in E^2$. Define $f, g : X \to X$ as follows:

$$\left.\begin{aligned} f(r, \theta) &= \left(\frac{r}{2 + \sin \theta}, \theta\right) \\ \text{and} \qquad g(r, \theta) &= \left(r, \frac{\pi}{2} - \frac{\pi/2 - \theta}{2}\right) \end{aligned}\right\}, 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \pi/2.$$

Then it can be easily verified that f and g are nonexpansive and

$$\left.\begin{aligned} &\text{(a)} \quad f^i g(r, \theta) - g f^i(r, \theta) \to 0 \\ &\text{(b)} \quad f^i g^i(r, \theta) - (fg)^i(r, \theta) \to 0 \\ &\text{(c)} \quad g^i f^i(r, \theta) - (gf)^i(r, \theta) \to 0 \end{aligned}\right\}, \text{ uniformly over } X \text{ as } i \to \infty$$

whereas, $g^i f(r, \theta) - f g^i(r, \theta) \not\to 0$ everywhere in X.

Example 3.3 — Let $X = \left\{0, 1, \frac{n}{n+1}; n = 1, 2, \ldots\right\}$. Then X is a closed subset in the space of reals.

Define $f, g : X \to X$ by the rule:

$$f\left(\frac{n}{n+1}\right) = \frac{n+1}{n+2}, \; n = 0, 1, 2, \ldots$$

$$f(1) = 1$$

$$g(0) = 1, \; g(\tfrac{1}{2}) = 0 \text{ and } g\left(\frac{n}{n+1}\right) = \frac{n}{n+1}, \; \forall \; n > 1.$$

Then f and g are nonexpansive and

$$f^i g(x) - g f^i(x) \to 0$$

uniformly over X as $i \to \infty$. However

$$g^i f(0) - f g^i(0) \not\to 0$$

and $$f^i g^i(0) - (fg)^i(0) \not\to 0 \text{ as } i \to \infty.$$

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