Indian J. pure appl. Math., 10(11) : 1337-1342, November 1979

THE $|E, q|$-SUMMABILITY OF A SERIES ASSOCIATED WITH FOURIER SERIES

PREM CHANDRA

School of Studies in Mathematics, Vikram University, Ujjain 456010

AND

VIRENDRA SINGH YADAVA

Hubba Lal Inter College, Agra 282001

(*Received* 5 *December* 1978; *after revision* 9 *April* 1979)

The authors investigate a sufficient condition for the absolute Euler summability of a series associated with Fourier series. They further show that the above condition is the best possible in a certain sense.

1. INTRODUCTION

Let $f(t)$ be 2π-periodic and L-integrable over $(-\pi, \pi)$ and let its Fourier series, at a point x, be

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n (x).$$

Essential supremum — A function f is called essentially bounded if there exists a positive, finite constant c such that $\{x / \, |f(x)| > c\}$ is a set of measure zero. The infimum of the values of c for which this statement is true is called the essential supremum of $|f|$, abbreviated to

$$\text{ess sup } |f| .$$

For fixed real numbers x and s, we write

$$\phi(t) = f(x + t) + f(x - t) - 2s \qquad \ldots(1.1)$$

and

$$s_n(x) = \tfrac{1}{2} a_0 + \sum_{m=1}^{n} A_m(x). \qquad \ldots(1.2)$$

Concerning the absolute Euler summability of Fourier series, a number of results have been obtained by Mohanty and Mohapatra (1968), Chandra (1972a, b, 1975, 1978) and Kwee (1972). In this paper, we first obtain the following theorem concerning $|E, q|$-summability $(q > 0)$ of a series related to a Fourier series:

Theorem 1 — Let

$$\int_0^\pi \frac{|\phi(t)|}{t} \log \frac{2\pi}{t}\, dt < \infty. \qquad \text{...(1.3)}$$

Then

$$\sum_{n=0}^{\infty} \frac{s_n(x) - s}{n+1} \in |E, q| \qquad (q > 0).$$

Our next attempt is to show that the condition (1.3) of Theorem 1 cannot be replaced by a lighter condition of the form:

$$\int_0^\pi \frac{|\phi(t)|}{t} \, |p(t)| \log \frac{2\pi}{t}\, dt < \infty \qquad \text{...(1.4)}$$

where the function $p(t)$ it defined in $(0, \pi)$ and that

$$p(t) = o(1), \quad \text{as} \quad t \downarrow 0. \qquad \text{...(1.5)}$$

It is known (see Lemma 1) that the inclusion:

$$|E, q| \subset |B| \qquad (q > -1)$$

is strict, therefore we now prove, in fact, a deeper result than what we have stated above. To be precise, we prove the following.

Theorem 2 — Let the function $p(t)$, defined in $(0, \pi)$, satisfy (1.5). Then (1.4) implies that

$$\sum_{n=0}^{\infty} \frac{s_n(x) - s}{n+1} \notin |B|. \qquad \text{...(1.6)}$$

We shall write throughout the paper

$$g(t) = \frac{\phi(t)}{\sin \frac{1}{2} t} \, p(t) \log \frac{2\pi}{t}.$$

2. Lemmas

We shall use the following lemmas in the proofs of the theorems:

Lemma 1 — If $q > -1$, then $|E, q| \subset |B|$.

This is due to Knopp and Lorentz (1949) for $q > 0$, and the general case was investigated by Chandra (1975).

Lemma 2 — Let F be a measurable function over $(0, \infty) \times (0, \infty)$ and $g \in L(0, \infty)$. Then

$$G(y) = \int_0^\infty F(y, t)\, g(t)\, dt$$

is defined almost everywhere for each $g \in L(0, \infty)$ and

$$\int_0^\infty | G(y) |\, dy < \infty$$

if and only if

$$\operatorname*{ess\,sup}_{0<t<\infty} \int_0^\infty | F(y, t) |\, dy \leqslant K.$$

This is due to Sunouchi and Tsuchikura (1952).

3. Proofs of the Theorems

Proof of Theorem 1 — We have (Titchmarsh 1949, p. 403)

$$s_n(x) - s = \frac{1}{\pi} \int_0^\pi \phi(t) \frac{\sin (n + \frac{1}{2})\, t}{2 \sin \frac{1}{2} t}\, dt.$$

The series $\sum_{n=0}^{\infty} \frac{s_n(x) - s}{n + 1} \in | E, q | \qquad (q > 0)$ if (Chandra 1975)

$$\sum_{n=0}^{\infty} (q + 1)^{-n-1} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{s_m(x) - s}{m + 1} \right| < \infty.$$

However, in view of (1.3), it will be sufficient for the proof of the theorem to show that

$$\sum_{n=0}^{\infty} \frac{(1 + q)^{-n-1}}{(n + 1)} \left| \sum_{m=0}^{n} \binom{n+1}{m+1} q^{n-m} \sin (m + \tfrac{1}{2})\, t \right|$$

$$= O\left\{\log \frac{2\pi}{t}\right\} \qquad \ldots(3.1)$$

uniformly in $0 < t < \pi$.

Now splitting up $\sum_{0}^{\infty}$ into $\sum_{0}^{T}$ and $\sum_{n>T}$, where T is the integral part of $(2\pi/t)^2$, we observe that

$$\sum_{n=0}^{T} = O\left(\log \frac{2\pi}{t}\right)$$

and

$$\sum_{n>T} < 1 + q + \left(\frac{t}{2\pi}\right)^2 \sum_{n=0}^{\infty} (1+q)^{-n-1} (1 + q^2 + 2q \cos t)^{(n+1)/2}$$

$$= O(1)$$

uniformly in $0 < t < \pi$.

This proves (3.1) and consequently it completes the proof of Theorem 1.

Proof of Theorem 2 — Without any loss of generality, we assume, for the proof of the theorem, that

$$(p(t))^{-1} = O\left(\log \frac{2\pi}{t}\right), \quad \text{as} \quad t \downarrow 0. \qquad \ldots(3.2)$$

The series

$$\sum_{n=0}^{\infty} \frac{s_n(x) - s}{n+1} \in |B|$$

if and only if (see Knopp and Lorentz 1949)

$$I = \frac{1}{2\pi} \int_0^{\infty} e^{-y} \left| \int_0^{\pi} \frac{dt}{p(t) \log (2\pi/t)} \sum_{n=0}^{\infty} \frac{y^n}{n!} \frac{\sin (n + \frac{3}{2}) t}{(n+2)} g(t) \right| dy < \infty.$$

Now, by Lemma 2 and hypothesis (1.4), I converges if and only if

$$\operatorname*{ess\,sup}_{0<t<\pi} \frac{1}{|p(t)| \log (2\pi/t)} \int_0^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!\,(n+2)} \sin (n + \tfrac{3}{2}) t \right| dy \leqslant K. \qquad \ldots(3.3)$$

However

$$\int_0^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{n!\,(n+2)} \sin (n + \tfrac{3}{2}) t \right| dy$$

$$\geqslant \int_1^{\infty} e^{-y} \left| \sum_{n=0}^{\infty} \frac{y^n}{(n+1)!} \sin (n + \tfrac{3}{2}) t \right| dy - O(1)$$

$$= \int_1^\infty y^{-1} e^{-y} \mid \exp(y \cos t) \sin \{\tfrac{1}{2} t + y \sin t\} - \sin \tfrac{1}{2} t \mid dy - O(1)$$

$$\geqslant \cos \tfrac{1}{2} t \int_0^\infty y^{-1} \exp\{-y(1 - \cos t)\} \mid \sin (y \sin t) \mid dy$$

$$- \sin \tfrac{1}{2} t \int_1^\infty y^{-1} \exp\{-y(1 - \cos t)\} \mid \cos (y \sin t) \mid dy - O(1)$$

$$= J_1(t) - J_2(t) - O(1), \text{ say.}$$

However, uniformly in $0 < t < \pi$,

$$J_2(t) \leqslant K \mid p(t) \mid \log \frac{2\pi}{t}$$

therefore for the proof of (3.3) it is necessary that

$$\operatorname*{ess\,sup}_{0<t<\pi} \frac{\cos \frac{1}{2} t}{\mid p(t) \mid \log (2\pi/t)} \int_0^\infty y^{-1} \exp\{-y(1 - \cos t)\} \mid \sin (y \sin t) \mid dy \leqslant K. \quad \ldots(3.4)$$

Now,

$$\int_0^\infty y^{-1} \exp(-2y \sin^2 \tfrac{1}{2} t) \mid \sin (y \sin t) \mid dy$$

$$= \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} u^{-1} \exp(-u \tan \tfrac{1}{2} t) \mid \sin u \mid du$$

$$\geqslant \frac{1}{\pi} \sum_{n=1}^\infty n^{-1} \exp(-n\pi \tan \tfrac{1}{2} t) \left| \int_{(n-1)\pi}^{n\pi} \sin u \, du \right|$$

$$= \frac{2}{\pi} \log \{1 - \exp(-\pi \tan \tfrac{1}{2} t)\}^{-1}$$

so that, whenever (1.4) holds,

$$\frac{2 \cos \frac{1}{2} t}{\pi \mid p(t) \mid \log (2\pi/t)} \log \{1 - \exp(-\pi \tan \tfrac{1}{2} t)\}^{-1} \to \infty$$

as $t \downarrow 0$. Thus (3.4) does not hold. Consequently, it proves the theorem.

Acknowledgement

The authors are thankful to the referee for suggesting to introduce the definition of " essential supremum" and write the proofs of the theorems in a more compact form.

REFERENCES

Chandra, P. (1972a). On the | E, q | -summability of Fourier series. *Nanta Math.*, **5**(2), 8-13.

——————— (1972b). Addendum to "On the | E, q | -summability of Fourier series". *Nanta Math.*, **5**(3), 8-9.

——————— (1975). On some summability methods. *Boll. Un. Mat. Ital.* (4), **12**(3), 211-24 [*Math. Rev.*, **54** (1977), 119-20].

——————— (1977). On the | E, q | -summability of a Fourier series and its conjugate series. *Riv. Mat. Univ. Parma* (4), **3**, 65-78.

——————— (1978). Absolute summability by (E, q) means. *Riv. Mat. Univ. Parma* (4), (to appear).

Knopp, K., and Lorentz, G. G. (1949). Beiträge zur-absoluten Limitierung. *Arch. Math.* (*Basel*), **2**, 10-16.

Kwee, B. (1972). The absolute Euler summability of Fourier series. *J. Austr. math. Soc.*, **13**, 129-40.

Mohanty, R., and Mohapatra, S. (1968). On the | E, q | summation of Fourier series and its conjugate series. *J. Indian math. Soc.* (*N.S.*), **32**, 131-40.

Sunouchi, G., and Tsuchikura, T. (1952). Absolute regularity for convergent integrals. *Tôhoku Math. J.* (2), **4**, 153-56.

Titchmarsh, E. C. (1949). The Theory of Functions. The University Press, Oxford.