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ON THE LOCALIZATION OF THE SUMMABILITY $|(R, \lambda_n, 1)(C, r)|$ OF THE rth DERIVED SERIES OF A FOURIER SERIES

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This paper contains two general results on the localization of the absolute summability of the rth derived series of Fourier series.

§1. Let $\{S_n\}$ be the sequence of partial sums of the series $\Sigma\, a_n$ and let $\{\mu_n\}$ be a sequence of positive numbers such that

$$\lambda_n = \sum_{\nu=0}^{n} \mu_\nu \to \infty \quad \text{as} \quad n \to \infty.$$

The sequence to sequence transformation

$$R_n = \frac{1}{\lambda_n} \sum_{\nu=0}^{n} \mu_\nu S_\nu$$

defines the Riesz means of the series $\Sigma\, a_n$ of type λ and order one. If the sequence $\{R_n\}$ is of bounded variation, that is to say if

$$\Sigma \mid R_n - R_{n-1} \mid < \infty \qquad \text{...(1.1)}$$

then the series $\Sigma\, a_n$ is said to be absolutely summable $(R, \lambda_n, 1)$ or summable $|R, \lambda_n, 1|$.

Let $\sigma_n^{(\alpha)}$ and $t_n^{(\alpha)}$ denote the nth Cesàro means of order α $(\alpha > -1)$ of the sequences $\{S_n\}$ and $\{na_n\}$ respectively. Let r be a positive integer and $T_n^{(r)}$ denote the nth Cesàro sum of order r of the sequence $\{na_n\}$ so that

$$t_n^{(r)} = T_n^{(r)} / A_n^{(r)}$$

where
$$A_n^{(r)} = \frac{(n+1)(n+2)\ldots(n+r)}{1.2.3.\ldots.r}.$$

When the condition (1.1) is satisfied with $\{S_n\}$ replaced by $\sigma_n^{(r)}$, the series $\Sigma\, a_n$ is said to be summable $|(R, \lambda_n, 1)(C, r)|$.

We write

$$\chi_n = \mu_n \lambda_n^{-1}, \; \triangle u_n = \triangle^1 u_n = u_n - u_{n-1},$$

$$\triangle^r u_n = \triangle \triangle^{r-1} u_n, \; r \geqslant 2.$$

The following identities are well known (Bosanquet 1945, Kogbetliantz 1925, 1931):

$$t_n^{(r)} = n(\sigma_n^{(r)} - \sigma_{n-1}^{(r)}) \qquad \ldots(1.2)$$

$$na_n = \triangle^r T_n^{(r)} \qquad \ldots(1.3)$$

$$\triangle^r u_n = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} u_{n-\nu}. \qquad \ldots(1.4)$$

Throughout the paper $f(t)$ is a periodic function with period 2π and integrable in the Lebesgue sense in $(0, 2\pi)$.

§2. It follows from a theorem of Hyslop (1940) that the summability $|C, \delta|$, $\delta > 2$ of the first derived series of a Fourier series at a given point depends only upon the behaviour of the generating function in the immediate neighbourhood of the point and is thus a local property.

Lal (1962) demonstrated that $|C, 2|$-summability of the derived series of a Fourier series is not necessarily a local property. Further extensions of this result were obtained by Bhatt (1963, 1967) and Lal (1978). In this paper we prove the following two theorems.

Theorem 1 — Let $\{\chi_n\}$ be a monotonic non-increasing sequence and $\{\epsilon_n\}$ be a sequence of positive numbers such that

(i) $\epsilon_n \chi_n$ decreases monotonically to zero, and

(ii) the series $\Sigma\, \epsilon_n \chi_n$ is divergent.

Then the summability $|(R, \lambda_n, 1)(C, r)|$ of the series $\Sigma\, \alpha_n \epsilon_n$, where α_n denotes the nth term of the rth derived series of Fourier series of $f(t)$, is not necessarily a local property of the generating function.

Theorem 2 — Let the sequences $\{\chi_n\}$ and $\{n^{-2}\chi_n^{-1}\}$ be monotonic non-increasing and $\Sigma\, n^{-1}\chi_n < \infty$. Then in order that the summability $|(R, \lambda_n, 1) . (C, r)|$ of the rth derived series of the Fourier series may depend upon the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point $t = x$, it is sufficient that when r is even

$$\Sigma \mid A_n(x) \mid \chi_n < \infty \qquad \ldots(2.1)$$

and when r is odd

$$\Sigma \mid B_n(x) \mid \chi_n < \infty. \qquad \ldots(2.2)$$

These theorems include as particular cases all the earlier known results (Bhatt 1963, 1967; Lal 1962, 1978; Saxena 1964) in this direction. It is also worth noting that the condition $\Sigma \mid A_n(x) \mid \chi_n < \infty$ in Theorem 2 is also necessary for the $|(R, \lambda_n, 1)(C, r)|$ summability of the rth derived series of Fourier series when r is even (cf. Lemma 3 below). A similar remark holds for the condition

$$\Sigma \mid B_n(x) \mid \chi_n < \infty$$

when r is odd.

§3. The following lemmas are pertinent to the proof of our theorems.

Lemma 1 — If

$$\Sigma \mid \sigma_n^{(r)} - \sigma_{n-1}^{(r)} \mid \chi_n < \infty$$

then

$$\sum \frac{\mid a_n \mid}{n^r} \chi_n < \infty.$$

PROOF : We note that by virtue of the identity (1.2), the condition

$$\Sigma \mid \sigma_n^{(r)} - \sigma_{n-1}^{(r)} \mid \chi_n < \infty$$

implies that

$$\Sigma \mid t_n^{(r)} \mid n^{-1}\chi_n < \infty.$$

By virtue of the identities (1.3) and (1.4) we have

$$\begin{aligned}\sum \frac{\mid a_n \mid}{n^r} \chi_n &= \sum \frac{\mid \Delta^r T_n^{(r)} \mid}{n^{r+1}} \chi_n \\ &\leqslant 2^r \sum \frac{\mid T_n^{(r)} \mid}{n^{r+1}} \chi_n \\ &= O\left(\sum \frac{\mid t_n^{(r)} \mid}{n} \chi_n \right) = O(1).\end{aligned}$$

Lemma 2 (Dikshit 1965, Lemma 3) — If the sequence $\{\chi_n\}$ is monotonic non-increasing, then the summability $\mid R, \lambda_n, 1 \mid$ of $\Sigma\, a_n$ implies the absolute convergence of the series $\Sigma\, a_n \chi_n$.

Lemma 3 — If the sequence $\{\chi_n\}$ is monotonic non-increasing, then the summability $|(R, \lambda_n, 1)(C, r)|$ of Σa_n implies the convergence of the series

$$\sum \frac{|a_n|}{n^r} \chi_n.$$

This lemma follows from Lemmas 1 and 2.

Lemma 4 — If the series

$$\Sigma \mid \sigma_n^{(r)} \mid \chi_n < \infty$$

then the series Σa_n is summable $|(R, \lambda_n, 1) . (C, r)|$.

This lemma follows easily from a known result (Dikshit 1965, Lemma 4).

Lemma 5 (Bhatt 1967) — If $\sigma_n^{(r)}(x)$ denotes the nth Cesàro mean of order r of the rth derived series of Fourier series and if r is even, then

$$\sigma_n^{(r)}(x) = \left(\frac{2}{\pi}\int_0^\delta \phi(t)\left\{\frac{d^r}{dt^r}\left(\frac{1}{A_n^{(r)}}\sum_{\nu=0}^{n} A_{n-\nu}^{(r-1)}\frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right)\right\}dt\right)$$
$$-\left(\frac{4}{\pi}\frac{(n+\frac{1}{2})^r}{A_n^{(r)}}\int_0^\delta \phi(t)\sin\tfrac{1}{2}t\,\frac{\sin(n+\frac{1}{2}+\frac{1}{2}r)t}{(2\sin\frac{1}{2}\delta)^{r+2}}dt\right)$$
$$+O\left(\sum{}' \frac{|A_\nu(x)|}{(n-\nu)^2}\right) + O(|A_n(x)|) + O\left(\frac{1}{n}\right)$$

where Σ' denotes the summation over $-\infty < \nu \leqslant -1$, $1 \leqslant \nu \leqslant n-1$ and $n+1 \leqslant \nu < \infty$, and $A_{-\nu}(x) = A_\nu(x)$.

Proof of Theorem 1 — In order to prove Theorem 1, we have to establish that if $0 < \alpha < \beta \leqslant 2\pi$, there exists a function summable over (α, β) and zero in the remainder of $(0, 2\pi)$ such that the series $\Sigma\, \alpha_n \epsilon_n$ is not summable $|(R, \lambda_n, 1)(C, r)|$.

Let r be even. By virtue of Lemma 3 it is sufficient to show that there exists a function $F(t)$ summable over (α, β), such that

$$\Sigma \left| \int_\alpha^\beta F(t) \cos nt\, \epsilon_n \chi_n\, dt \right| = \infty.$$

Now for $0 < t < 2\pi\ (t \neq \pi)$, we have

$$\begin{aligned}\Sigma \mid \cos nt \mid \epsilon_n\chi_n &\geqslant \Sigma \cos^2 nt\, \epsilon_n\chi_n\\ &= \tfrac{1}{2}\Sigma(1+\cos 2nt)\,\epsilon_n\chi_n\\ &\geqslant \tfrac{1}{2}\Sigma\,\epsilon_n\chi_n - \tfrac{1}{2}\mid \Sigma \cos 2nt\,\epsilon_n\chi_n \mid = \infty\end{aligned}$$

and the theorem for the case when r is even follows from a result due to Bosanquet and Kestelman (1939, Theorem 1).

A similar proof holds for the case when r is odd.

Proof of Theorem 2 — Let us suppose that r is even. By Lemma 5 we have

$$\sigma_n^{(r)}(x) = \left(\frac{2}{\pi}\int_0^\delta \phi(t)\left\{\frac{d^r}{dt^r}\left(\frac{1}{A_n^{(r)}}\sum_{\nu=0}^{n} A_{n-\nu}^{(r-1)}\frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right)\right\}dt\right)$$

$$-\left(\frac{4}{\pi}\frac{(n+\frac{1}{2})^r}{A_n^{(r)}}\int_0^\delta \phi(t)\sin\tfrac{1}{2}t\,\frac{\sin(n+\frac{1}{2}+\frac{1}{2}r)t}{(2\sin\frac{1}{2}\delta)^{r+2}}\,dt\right)$$

$$+O\left(\sum{}'\frac{|A_\nu(x)|}{(n-\nu)^2}\right)+O(|A_n(x)|)+O\left(\frac{1}{n}\right)$$

$$=\sum_{k=1}^{5} M_n^{(k)}, \text{ say.}$$

We observe that for positive δ, however small but fixed, the convergence of the series $\Sigma\,|M_n^{(k)}|\,\chi_n$ $(k = 1, 2)$ depends only upon the behaviour of the generating function $f(t)$ in the immediate neighbourhood of the point x. Also, since by the hypotheses of the theorem

$$\Sigma\, M_n^{(4)}\chi_n = \Sigma\,|A_n(x)|\,\chi_n < \infty$$

and $$\Sigma\, M_n^{(5)}\chi_n = \Sigma\, n^{-1}\chi_n < \infty$$

by an appeal to Lemma 4, it follows that for establishing the theorem we need only show that

$$\sum_n M_n^{(3)}\chi_n = \sum_n \chi_n \sum_\nu{}'\frac{|A_\nu(x)|}{(n-\nu)^2}$$

is convergent.

We have

$$\sum_n \chi_n \sum_\nu{}'\frac{|A_\nu(x)|}{(n-\nu)^2}$$

$$=\sum_n \chi_n\left(\sum_{\nu=-\infty}^{-1}+\sum_{\nu=1}^{n-1}+\sum_{\nu=n+1}^{m+n}+\sum_{\nu=m+n+1}^{\infty}\right)\frac{|A_\nu(x)|}{(n-\nu)^2}$$

(*equation continued on p.* 1348)

$$= \sum_{n=2}^{\infty} \chi_n(P_1 + P_2 + P_3 + P_4), \text{ say.}$$

Now

$$\sum_{n=2}^{\infty} \chi_n P_1 \leqslant \sum_{n=1}^{\infty} \chi_n \sum_{\nu=n}^{\infty} \frac{|A_{\nu-n}(x)|}{\nu^2}$$

$$= \sum_{n=1}^{\infty} \chi_n \left\{ \sum_{\nu=n}^{2n-1} + \sum_{\nu=2n}^{\infty} \right\} \frac{|A_{\nu-n}(x)|}{\nu^2}$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{\nu=n}^{2n-1} |A_{\nu-n}(x)| \chi_{\nu-n}$$

$$+ \sum_{n=1}^{\infty} \chi_n \sum_{\nu=2n}^{\infty} \frac{|A_{\nu-n}(x)|}{(\nu-n)^2} \frac{\chi_{\nu-n}}{\chi_{\nu-n}}$$

$$\leqslant K + \sum_{n=1}^{\infty} \frac{\chi_n}{n^2 \chi_n} \sum_{\nu=2n}^{\infty} |A_{\nu-n}(x)| \chi_{\nu-n}$$

$$< \infty$$

by the hypotheses of the theorem.

And

$$\sum_{n=2}^{m} \chi_n P_2 = \sum_{n=2}^{m} \chi_n \sum_{k=1}^{n-1} \frac{|A_{n-k}(x)|}{k^2}$$

$$= \sum_{k=1}^{m-1} \frac{1}{k^2} \sum_{n=k+1}^{m} |A_{n-k}(x)| \chi_{n-k} \frac{\chi_n}{\chi_{n-k}}$$

$$< \infty$$

for by the hypotheses of the theorem $\chi_n/\chi_{n-k} \leqslant 1$ for $n \geqslant k+1$, and

$$\Sigma |A_n(x)| \chi_n < \infty.$$

$$\sum_{n=2}^{m} \chi_n P_3 \leqslant \sum_{n=1}^{m} \chi_n \sum_{k=1}^{n} \frac{|A_{n+k}(x)|}{k^2} + \sum_{n=1}^{m} \chi_n \sum_{k=n}^{m} \frac{|A_{n+k}(x)|}{k^2}$$

$$= \sum_{k=1}^{m} \frac{1}{k^2} \sum_{n=k}^{m} |A_{n+k}(x)| \chi_{n+k} \frac{\chi_n}{\chi_{n+k}}$$

$$+ \sum_{n=1}^{m} \chi_n \sum_{k=n}^{m} \frac{|A_{n+k}(x)|}{(n+k)^2} \frac{\chi_{n+k}}{\chi_{n+k}} \left(\frac{n+k}{k}\right)^2$$

$$= O\left(\sum_{k=1}^{m} \frac{1}{k^2} \sum_{n=k}^{m} |A_{n+k}(x)| \chi_{n+k} \frac{\chi_n}{\chi_{2n}}\right)$$

$$+ O\left(\sum_{n=1}^{m} \chi_n \frac{1}{n^2 \chi_{2n}} \sum_{k=n}^{m} |A_{n+k}(x)| \chi_{n+k}\right)$$

$$= O\left(\sum_{k=1}^{m} \frac{1}{k^2} \sum_{n=k}^{m} |A_{n+k}(x)| \chi_{n+k}\right) + O(1)$$

$$= O(1)$$

under the hypotheses of the theorem and the fact that $\chi_n = O(\chi_{2n})$ which follows by the condition that the sequence $\{n^{-2}\chi_n^{-1}\}$ is monotonic non-increasing.

Finally

$$\sum_{n=2}^{m} \chi_n P_4 \leqslant \sum_{n=1}^{m} \chi_n \sum_{\nu=m+n+1}^{\infty} \frac{|A_\nu(x)|}{(n-\nu)^2}$$

$$= O\left(\frac{1}{m}\right) \sum_{n=1}^{m} \chi_n = O(1), \text{ as } m \to \infty.$$

This completes the proof of the theorem.

References

Bhatt, S. N. (1963). An aspect of local property of absolute summability of the derived Fourier series. *Math. Z.*, **80**, 384-89.

——— (1967). An aspect of local property of the absolute summability of the *r*th derived series. *Indian J. Math.*, **9**, 17-24.

Bosanquet, L. S. (1945). Note on convergence and summability factors. *J. Lond. math. Soc.*, **20**, 39-48.

Bosanquet, L. S., and Kestelman, H. (1939). The absolute convergence of series of integrals. *Proc. Lond. math. Soc.* (2), **45**, 88-97.

Dikshit, G. D. (1965). Localization relating to the summability $| R, \lambda_n, 1 |$ of Fourier series. *Indian J. Math.*, **7**, 31-39.

Hyslop, J. M. (1940). On the absolute summability of the successively derived series of a Fourier series and its allied series, *Proc. Lond. math. Soc.* (2), **46**, 55-80.

Kogbetliantz, E. (1925). Sur les series absolument sommables par la methode des moyennes arithmetiques. *Bull. Sci. Math.* (2), **49**, 234-56.

——— (1931). Sommation des series et integrales divergentes par les moyennes arithmetiques et typiques. *Mem. Sci. Math.* No. 51, Paris.

Lal, S. N. (1962). An aspect of local property of the $| C, 2 |$ summability of the derived series of Fourier series. *Ann. Mat. pura applic.*, **59**, 65-75.

——— (1978). On the localisation of the absolute summability of the rth derived series of Fourier series. *Math. Chronicle*, **7**, 48-56.

Saxena, A. (1964). An aspect of local property of absolute summability of the derived series of a Fourier series. *Rend. Circ. Mat. Palermo*, **13**, 263-72.