*Indian J. pure appl. Math.*, **10**(11) : 1351-1361, November 1979

# FLOW BETWEEN A TORSIONALLY OSCILLATING IMPERMEABLE DISC AND A STATIONARY NATURALLY PERMEABLE DISC

P. D. VERMA AND DILEEP SINGH CHAUHAN
*Department of Mathematics, University of Rajasthan, Jaipur* 302004

(*Received* 28 *December* 1978; *after revision* 24 *March* 1979)

In the present investigation we study the flow induced in a viscous incompressible fluid from small torsional oscillations of an impermeable infinite disc bounded coaxially at a distance $d$ by an another stationary naturally permeable infinite disc of finite thickness $l$. It is found that the steady radial velocity increases in magnitude with the increase of Reynolds number and $\beta(= k^*/d^2)$, where $k^*$ is permeability. There is slip in the steady radial velocity at the porous boundary and the same increases with the increase of Reynolds number and $\beta$. Typical streamlines of the steady part of the radial axial flow are shown graphically for various values of flow parameters and discussed.

## 1. INTRODUCTION

The flow induced by an infinite disk performing torsional oscillations in a viscous incompressible fluid has been studied by Rosenblat (1959). He obtained the solution of Navier-Stokes' equations in a power series in $\epsilon(= \Omega/w)$, the amplitude of oscillation. Rosenblat (1960) further investigated the flow induced in a viscous fluid from small torsional oscillations of two infinite discs. He studied the following two cases :

(i) when one disc is oscillating and the other is at rest, and

(ii) when both discs oscillate with the same amplitude and frequency, but with a phase difference of 180°.

In the present paper we studied the flow induced in a viscous incompressible fluid from small torsional oscillations of two infinite discs, when one disc, which is impermeable, is performing small torsional oscillations, while the other is at rest and have a permeable thickness of height $l$. The method adopted for analysis is similar to that of Rosenblat (1960). In free fluid region the Navier-Stokes' equations govern the flow, while in porous region, the flow is governed by Darcy's equations. We applied the no-slip conditions at impermeable disc and the slip conditions suggested by Beavers and Joseph (1967) at the porous boundary. The steady radial velocity and the streamline patterns are drawn for various values of flow parameters.

## 2. Equations of Motion

We consider a viscous incompressible fluid confined between two coaxial discs of infinite radius placed at a distance $d$. The cylindrical polar coordinates $(r, \theta, z)$ are being used with the origin at the centre of the lower disc and $z$-axis normal to it. The lower disc $z = 0$ is impermeable and performs small torsional oscillations about the common axis of frequency $n$ and angular speed $\Omega$, while the upper disc $z = d$, has a lining of a naturally permeable material of thickess $l$ with an impermeable surface at the top, and remains at rest.

The flow in the free fluid region $[0 \leqslant z \leqslant (d - l)]$ is governed by the following Navier-Stokes' equations :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right] \quad ...(2.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right] \quad ...(2.2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right] \quad ...(2.3)$$

and the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0. \quad ...(2.4)$$

The Darcy's law holds good in the porous region. The flow in the porous region $[(d - l) \leqslant z \leqslant d]$ is governed by the Darcy's equations :

$$U = -\frac{k^*}{\mu}\frac{\partial P}{\partial r} \quad ...(2.5)$$

$$V = 0 \quad ...(2.6)$$

$$W = -\frac{k^*}{\mu}\frac{\partial P}{\partial z} \quad ...(2.7)$$

and

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z} = 0. \quad ...(2.8)$$

where $k^*$ is the permeability of the material and $\mu$ the coefficient of viscosity.

The boundary conditions are

at $z = 0$,

$$u = w = 0, \; v = r\Omega e^{int} \quad \ldots(2.9)$$

at $z = (d - l)$,

$$p = P, \; w = W, \; e_{r\theta} = \gamma v, \; e_{rz} = \gamma(u - U) \quad \ldots(2.10)$$

where

$$e_{r\theta} = r \frac{\partial}{\partial r}\left(\frac{v}{r}\right), \; e_{rz} = \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right), \; \gamma = \frac{\alpha}{\sqrt{k^*}}$$

and $\alpha$ is a constant depending upon the porous material.

At $z = d$,

$$W = 0. \quad \ldots(2.11)$$

Let us put

$$\left.\begin{aligned} & v = r\Omega e^{i\tau} g(y), \; u = \frac{r\Omega^2}{n}\frac{\partial F}{\partial y}(y, \tau), \; w = -2d\frac{\Omega^2}{n}F(y, \tau) \\ & \frac{p}{\rho} = \Omega^2 d^2 p_0(y, \tau) + \tfrac{1}{2}\Omega^2 r^2 K(\tau), \; y = z/d, \; \tau = nt \end{aligned}\right\} \quad \ldots(2.12)$$

and

$$\left.\begin{aligned} & U = \frac{r\Omega^2}{n}\frac{\partial \bar{F}}{\partial y}(y, \tau), \; W = -2d\frac{\Omega^2}{n}\bar{F}(y, \tau), \; V = r\Omega e^{i\tau}\bar{g}(y) \\ & \frac{P}{\rho} = \Omega^2 d^2 P_0(y, \tau) + \tfrac{1}{2}\Omega^2 r^2 \bar{K}(\tau). \end{aligned}\right\} \quad \ldots(2.13)$$

Substituting (2.12) and (2.13) in (2.1) - (2.11), we get

$$\frac{\partial^2 F}{\partial y \partial \tau} + \left(\frac{\Omega}{n}\right)^2\left[\left(\frac{\partial F}{\partial y}\right)^2 - 2F\frac{\partial^2 F}{\partial y^2}\right] - (ge^{i\tau})^2 = -K(\tau) + \frac{1}{R}\frac{\partial^3 F}{\partial y^3} \quad \ldots(2.14)$$

$$ig + 2\left(\frac{\Omega}{n}\right)^2\left(g\frac{\partial F}{\partial y} - F\frac{\partial g}{\partial y}\right) = \frac{1}{R}\frac{\partial^2 g}{\partial y^2} \quad \ldots(2.15)$$

$$2\frac{\partial F}{\partial z} - 4\left(\frac{\Omega}{n}\right)^2 F\frac{\partial F}{\partial y} = \frac{\partial p_0}{\partial y} + \frac{2}{R}\frac{\partial^2 F}{\partial y^2} \quad \ldots(2.16)$$

$$\frac{\partial \bar{F}}{\partial y} = -\beta R\bar{K} \quad \ldots(2.17)$$

$$\bar{g} = 0 \quad ...(2.18)$$

$$2\bar{F} = \beta R \frac{\partial P_0}{\partial y} \quad ...(2.19)$$

and the boundary conditions become

at $y = 0$,

$$F = \frac{\partial F}{\partial y} = 0, \ g = 1 \quad ...(2.20)$$

at $y = (1 - \epsilon)$,

$$p_0 = P_0, \ K = \bar{K}, \ F = \bar{F}, \ g = 0$$

$$\frac{\partial^2 F}{\partial y^2} = \gamma d \left( \frac{\partial F}{\partial y} - \frac{\partial \bar{F}}{\partial y} \right) \quad ...(2.21)$$

and at $y = 1$, $\bar{F} = 0$ ...(2.22)

where $R = \frac{nd^2}{\nu}$, $\beta = \frac{k^*}{d^2}$ and $\epsilon = \frac{l}{d}$ $(0 < \epsilon < 1)$.

## 3. Solutions

*Transverse Component*

Following the method of analysis adopted by Rosenblat (1960), the transverse component of velocity in free fluid for small Reynolds number, is given by, in real notation,

$$\frac{v}{r\Omega} = \left[ 1 - \frac{y}{(1 - \epsilon)} \right] \left[ 1 - \frac{R^2 y}{360} (8 + 8y - 12y^2 + 3y^3 - 24\epsilon + 24\epsilon^2 - 8\epsilon^3 - 16\epsilon y + 12y^2\epsilon + 8\epsilon^2 y) \right] \cos nt + \left[ \frac{Ry}{6} (2 - 2\epsilon - y) \right] \sin nt + O(R^3). \quad ...(3.1)$$

with the amplitude approximately,

$$\frac{|v|}{r\Omega} \approx \left[ 1 - \frac{y}{(1 - \epsilon)} \right] \left[ 1 - \frac{R^2 y (2 - 2\epsilon - y)}{180} (2 - 2y + y^2 - 4\epsilon + 2\epsilon^2 + 2\epsilon y) \right] \quad ...(3.2)$$

and phase angle $\approx \tan^{-1} Ry(2 - 2\epsilon - y)/6$. ...(3.3)

This is same as given by Rosenblat (1960) when the dimensionless thickness of porous material ($\epsilon$) becomes zero.

*Radial-axial Component*

The governing equations indicate that the radial-axial flow has a mean steady component and a fluctuating component of frequency twice that of the oscillating plate, as mentioned by Rosenblat (1960). Thus we take

$$F(y, \tau) = \text{Re}\,[f(y) + h(y)\, e^{2i\tau}] \qquad \text{...(3.4)}$$

and

$$K(\tau) = \text{Re}\,[K_0 + K_1 e^{2i\tau}]. \qquad \text{...(3.5)}$$

The mean steady radial-axial component is given by

$$f(y) = A_3 + A_2 y + \frac{1}{2} A_1 y^2 + \frac{1}{12} K_0 \lambda^2 y^3 + \frac{1}{4B\lambda}\,[\sinh \lambda(a-y) + \sin \lambda(a-y)] \qquad \text{...(3.6)}$$

$$f'(y) = A_1 y + A_2 + \frac{1}{4} K_0 \lambda^2 y^2 - \frac{1}{4B} \times [\cosh \lambda(a-y) + \cos \lambda(a-y)] \qquad \text{...(3.7)}$$

where

$$A_1 = \frac{2}{a^2}\left\{-(A_3 + aA_2) - K_0\left[\frac{1}{12} a^3 \lambda^2 - \frac{1}{2} \beta\lambda^2(1-a)\right]\right\}$$

$$A_2 = \frac{1}{4B}\,[\cosh \lambda a + \cos \lambda a], \quad A_3 = -\frac{1}{4B\lambda}(\sinh \lambda a + \sin \lambda a)$$

$$A_4 = \left[-\frac{1}{3} a^3 - \beta(1-a) + \frac{1}{12}\gamma d a^4 - \frac{1}{2}\gamma d\beta a^2 + \gamma d\beta a\right]$$

$$K_0 = \frac{1}{\lambda^2 A_4}\left[-2(1-\gamma d a) A_3 - (2a - \gamma d a^2) A_2 + \frac{1}{2B}\gamma d a^2\right]$$

$$B = (\cosh \lambda a - \cos \lambda a) \quad \text{and} \quad a = (1-\epsilon).$$

For small Reynolds number, we have

$$f(y) \approx \frac{R}{360 A_4}\left\{-y^2(-24a + 9\gamma d a^2)\left[\frac{a^3}{12} - \frac{1}{2}\beta(1-a)\right] + \frac{y^3}{12}(-24a^3 + 9\gamma d a^4) + \frac{A_4}{a^2}(18y^2a^3 - 30y^3a^2 + 15ay^4 - 3y^5)\right\} \qquad \text{...(3.8)}$$

$$f'(y) \approx \frac{R}{360 A_4}\left\{-y(-24a + 9\gamma d a^2)\left[\frac{1}{6} a^3 - \beta(1-a)\right] + \right.$$

(*equation continued on p.* 1356)

$$+ \frac{y^2}{4} [(-24a^3 + 9\gamma d a^4)]$$

$$- \frac{A_4}{a^2} (-36a^3 y + 90a^2 y^2 - 60ay^3 + 15y^4)\Big\} \quad ...(3.9)$$

and

$$K_0 \approx \frac{1}{720A_4} (-24a^3 + 9\gamma d a^4). \quad ...(3.10)$$

The time-dependent component of the radial-axial flow is given by,

$$h = \Big[B_1 + B_2 (\sinh \lambda sy + \cosh \lambda sy) + B_3 (\cosh \lambda sy - \sinh \lambda sy)$$

$$+ i\left(\frac{K_1}{2} + \frac{1}{4b}\right) y - \frac{c}{8b\lambda} \sinh \lambda(a - y)\, c\Big] \quad ...(3.11)$$

$$h' = \Big[\frac{1}{\lambda s} B_2 (\cosh \lambda sy + \sinh \lambda sy) + \frac{1}{\lambda s} B_3 (\sinh \lambda sy - \cosh \lambda sy)$$

$$+ i\left(\frac{K_1}{2} + \frac{1}{4b}\right) + \frac{1}{8b\lambda^2} \cosh \lambda(a - y)\, c\Big] \quad ...(3.12)$$

and

$$K_1 = M/N \text{ where}$$

$$M = \Big\{\Big[(1 - \cosh \lambda as + \sinh \lambda as)(1 + \cosh \lambda ca)$$

$$- \lambda s\left(\frac{1-i}{2\lambda} \sinh \lambda ac + a\right)\Big] [(\lambda^2 i + \gamma d\lambda s)(\cosh \lambda as - \sinh \lambda as)$$

$$+ (\lambda^2 i - \gamma d\lambda s)(\sinh \lambda as + \cosh \lambda as)] - [2(1 - \cosh \lambda as)]$$

$$\times \Big[(1 + \cosh \lambda ac)(\lambda^2 i + \gamma d\lambda s)(\cosh \lambda as - \sinh \lambda as \frac{\lambda si}{2b}\Big]\Big\}$$

$$N = 2b\, \{\lambda s(\beta\lambda^2 \epsilon i + a) - (1 - \cosh \lambda as + \sinh \lambda as)][(\lambda^2 i + \gamma d\lambda s)$$

$$\times (\cosh \lambda as - \sinh \lambda as) + (\lambda^2 i - \gamma d\lambda s)(\sinh \lambda as + \cosh \lambda as)]$$

$$- [2(1 - \cosh \lambda as)] [\lambda s(\gamma d - \gamma d\beta\lambda^2 i) - (\lambda^2 i + \gamma d\lambda s)$$

$$\times (\cosh \lambda as - \sinh \lambda as)]\}$$

$$B_1 = \Big[\frac{c}{8b\lambda} \sinh \lambda ac - (B_2 + B_3)\Big]$$

$$B_2 = \frac{\{[\lambda s((c/2\lambda) \sinh \lambda ac + a) - i(1 - \cosh \lambda as + \sinh \lambda as)] + 2ib\, [\lambda s(\beta\lambda^2 \epsilon i + a) - (1 - \cosh \lambda as + \sinh \lambda as)]\, K_1\}}{8\lambda sb(1 - \cosh \lambda as)}$$

$$B_3 = \frac{1}{\lambda s}\left[\lambda s B_2 + \frac{iK_1}{2} + \frac{i}{4b}(1 + \cosh \lambda ac)\right]$$

$$s = \frac{1}{\sqrt{2}}(1 + i), \quad c = (1 + i), \quad b = (\cosh \lambda ac - 1).$$

Similarly, we have the solutions in porous region of the form :

$$\bar{F}(y, \tau) = \text{Re}\,[\bar{f}(y) + \bar{h}(y)\, e^{2i\tau}]. \qquad \text{...(3.13)}$$

Putting this in (2.17), and equating the coefficient of $e^{2i\tau}$ and independent of it, we get

$$\frac{\partial \bar{f}}{\partial y} = -\,\beta R K_0 \qquad \text{...(3.14)}$$

and

$$\frac{\partial \bar{h}}{\partial y} = -\,\beta R K_1 \qquad \text{...(3.15)}$$

with the boundary conditions,

$$\bar{f} = \bar{f}' = 0, \ \bar{h} = \bar{h}' = 0 \ \text{at} \ y = 1.$$

On solving eqns. (3.14) and (3.15) subject to these boundary conditions, we have

$$\bar{f} = \beta R K_0 (1 - y) \qquad \text{...(3.16)}$$

and

$$\bar{h} = \beta R K_1 (1 - y). \qquad \text{...(3.17)}$$

The stream functions for steady radial-axial flow for the free fluid and for the porous region are given by

$$\Psi_1 = \frac{\psi_1}{(d^5\Omega^2/\nu)} = \frac{(dr^2\Omega^2/n)\, f(y)}{(d^5\Omega^2/\nu)} = \frac{\bar{r}^2}{R}\, f(y) \qquad \text{...(3.18)}$$

$$\Psi_2 = \frac{\psi_2}{(d^5\Omega^2/\nu)} = \frac{(dr^2\Omega^2/n)\, \bar{f}(y)}{(d^5\Omega^2/\nu)} = \frac{\bar{r}^2}{R}\, \bar{f}(y) \qquad \text{...(3.19)}$$

where $\quad \bar{r} = r/d.$

## 4. Numerical Discussion

The steady radial velocity profiles (Figs. 1, 2) are drawn for different values of Reynolds number $R$ and $\beta$, keeping the dimensionless thickness of porous material $\epsilon = 0.2$. The radial velocity ($f'$) is zero at $y = 0.48$, for $\beta = 0.2$, $\alpha = 1.45$, $\epsilon = 0.2$ and Reynolds number is taken to be small. The steady radial velocity increases in magnitude with the increase of $R$ and $\beta$ near both the discs. There is slip in the steady radial velocity at the porous disc, which can be seen by the discontinuity at $y = 0.8$ in Figs. 1 and 2. The slip at $y = 0.8$ increases with the increase of $R$ and $\beta$. The steady radial velocity changes sign at a greater distance from the oscillating disc with the increase of $\alpha$ while with the increase of $\beta$ it changes sign at a lesser distance from the oscillating disc.

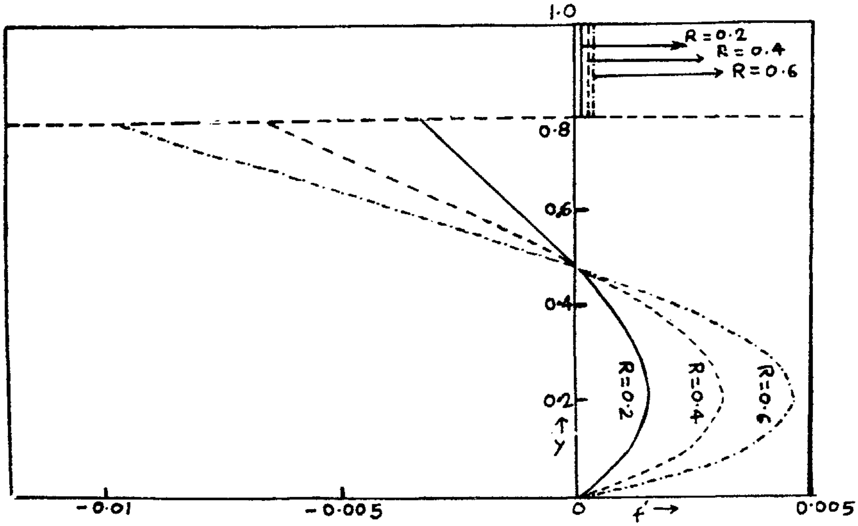


FIG. 1. Steady radial velocity $f'$ versus $y$ for $\epsilon = 0.2$, $\beta = 0.2$, $\alpha = 1.45$.

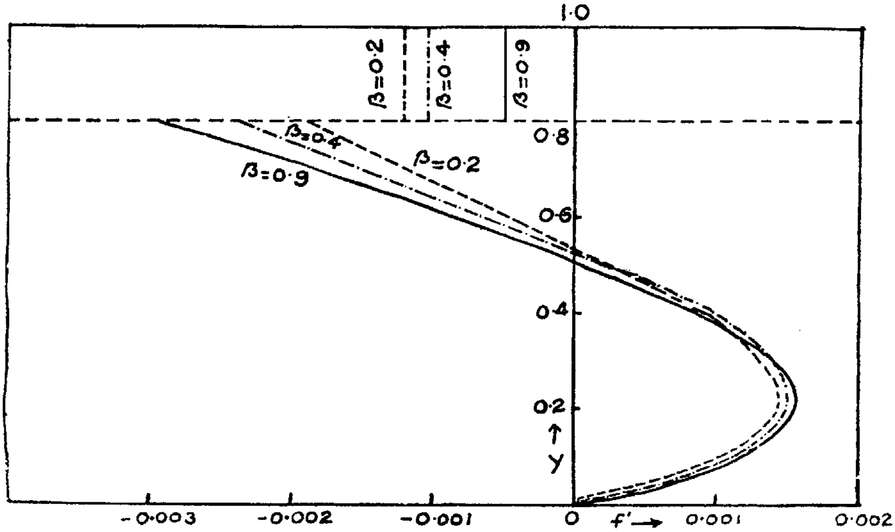


FIG. 2. Steady radial velocity $f'$ versus $y$ for $\epsilon = 0.2$, $R = 0.2$, $\alpha = 4.0$.

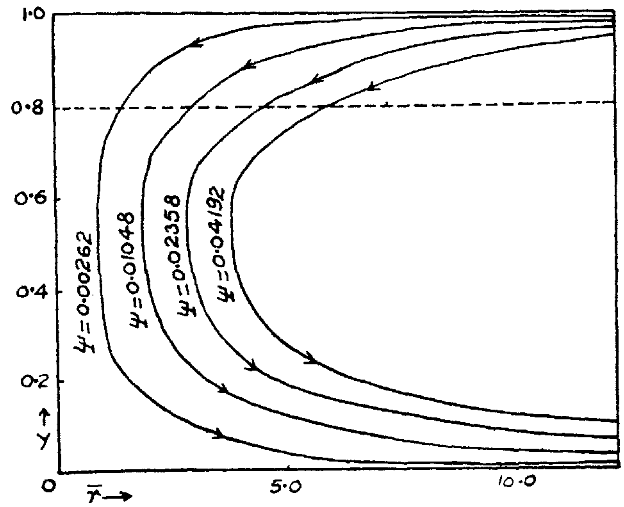


FIG. 3. Typical streamlines of steady radial axial flow for $R = 0.5$, $\varepsilon = 0.2$, $\beta = 0.4$, $\alpha = 4$.

Figure 3 depicts schematically, typical streamlines of steady radial-axial flow for $R = 0.5$, $\beta = 0.4$, $\alpha = 4$ and $\epsilon = 0.2$. Due to the oscillation of the impermeable disc, the fluid is thrown radially outwards near it and a pressure gradient is developed at a large distance from the axis of symmetry causing inward flow in the porous material. The streamlines are also drawn in Figs. 4, 5 and 6 for $R = 0.2$, $\beta = 0.2$, $\epsilon = 0.2$ and $\alpha = 1$, 1.45, 1.49 respectively. It is interesting to note that for $\alpha = 1$ in Fig. 4, the streamline $\Psi = 0$ divides the free fluid region showing the fluid which is thrown out radially near the oscillating disc will develop a pressure gradient near the mid plane causing the streamlines for negative values of $\Psi$ piercing the porous material. In Fig. 5 for $\alpha = 1.45$, the streamline $\Psi = 0$ is shifted very near to the porous boundary. In Fig. 6 for $\alpha = 1.49$, the streamline $\Psi = 0$ coincides with the porous boundary and there is no flow in the porous material.

If we examine the effect in the flow due to variation of the thickness of free fluid region given by $a = (1 - \epsilon)$, then the boundary of the porous material will act as an impermeable wall for

$$a = \frac{8\sqrt{\beta}}{3\alpha}, \text{ provided } \beta \neq \frac{a^3}{3(15 - a)}.$$

It may find applications in the situations where the fluid from the free fluid region is not required to enter the porous material, it can be achieved by adjusting the thickness of the porous material.

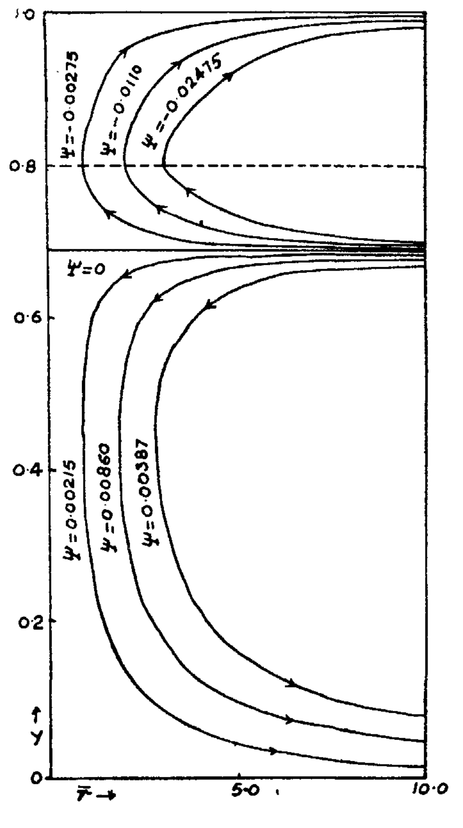


FIG. 4. Typical streamlines of steady radial axial flow for $R = 0.2$, $\beta = 0.2$, $\varepsilon = 0.2$, $\alpha = 1.0$.

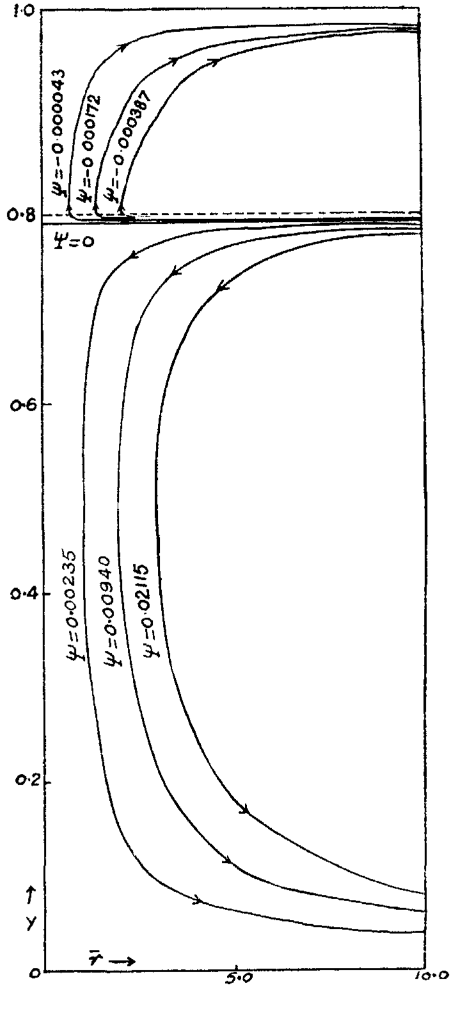


FIG. 5. Typical streamlines of steady radial axial flow for $R = 0.2$, $\beta = 0.2$, $\varepsilon = 0.2$, $\alpha = 1.45$.

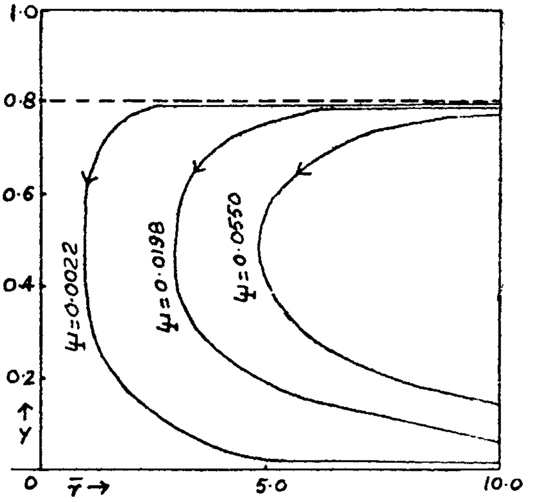


FIG. 6. Typical streamlines of steady radial axial flow for $R = 0.2$, $\beta = 0.2$, $\varepsilon = 0.2$, $\alpha = 1.49$.

ACKNOWLEDGEMENT

The authors are thankful to the referee for valuable suggestions. One of them (D.S.C.) is thankful to the Council of Scientific and Industrial Research for awarding a Junior Research Fellowship.

REFERENCES

Beavers, G. S., and Joseph, D. D. (1967). Boundary conditions at a naturally permeable wall. *J. Fluid Mech.*, **30**, 197.

Rosenblat, S. (1959). Torsional oscillations of a plane in a viscous fluid. *J. Fluid Mech.*, **6**, 206.

——— (1960). Flow between torsionally oscillating disks. *J. Fluid Mech.*, **8**, 388.