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COMMON FIXED POINT THEOREMS FOR MAPPINGS SATISFYING RATIONAL INEQUALITIES

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In this paper common fixed point theorems have been established for a new class of contractive type mappings satisfying rational inequalities in complete metric spaces.

1. INTRODUCTION

Let (X, d) be a complete metric space, and let $T : X \to X$ satisfy

$$d(Tx, Ty) \leqslant \alpha d(x, y)$$

for all x, y in X and $0 \leqslant \alpha < 1$. By Banach's (1922, p. 160) fixed point theorem T has a unique fixed point. Acccording to Ćirić's (1974a) fixed point theorem the following condition also implies that T has a unique fixed point:

$$d(Tx, Ty) \leqslant \alpha \max \{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2} [d(x, Ty) + d(y, Tx)]\}$$

for all x, y in X and $0 \leqslant \alpha < 1$.

During the past few years these results have been generalized and unified in several directions by many authors, of which we mention those of Ćirić (1972, 1974b), Matkowski (1977), Pachpatte (1976, 1977a, b), Pal and Maiti (1977) and Rhoades (1977). Recently, Pachpatte (1977a, 1978), Ćirić (1976), Fisher (1977) and Kasahara (1978) have established some common fixed point theorems for contractive type conditions satisfying rational inequalities. Our objective here is to establish some common fixed point theorems for a new class of contractive type mappings satisfying rational inequalities in complete metric spaces.

2. MAIN RESULTS

In this section we state and prove our main results on the common fixed points of self-mappings of a complete metric space satisfying rational inequalities.

Our main result is established in the following theorem.

Theorem 1 — Let S and T be mappings of a nonempty complete metric space X into itself satisfying the inequality

$$d(Sx, Ty) \leqslant \frac{q \max \{[d(x, y)]^2, [d(x, Sx)]^2, [d(y, Ty)]^2, \frac{1}{2}[d(x, Ty)]^2, \frac{1}{2}[d(y, Sx)]^2\}}{d(x, Sx) + d(y, Ty)} \quad \ldots(1)$$

for all x, y in X for which $d(x, Sx) + d(y, Ty) \neq 0$, $q \in (0, 1)$. Then S and T have a common fixed point z. Further, if $d(x, Sx) + d(y, Ty) = 0$ implies that $d(Sx, Ty) = 0$, then z is the unique fixed point of S and T.

PROOF: Let $x_0 \in X$ be arbitrary and define a sequence

$$x_{2n+1} = Sx_{2n}, \; x_{2n+2} = Tx_{2n+1}, \; n = 0, 1, 2, \ldots . \quad \ldots(2)$$

Suppose first of all that $d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) = 0$ for some n. Then it follows immediately that $z = x_{2n}$ is a common fixed point of S and T. Similarly, $d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2n+3}) = 0$ for some n implies that $z = x_{2n+1}$ is a common fixed point of S and T.

Now suppose that $d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \neq 0$ and $d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2n+3}) \neq 0$ for $n = 0, 1, 2, \ldots$. Then

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leqslant \frac{q \max \{[d(x_{2n}, x_{2n+1})]^2, [d(x_{2n}, Sx_{2n})]^2, [d(x_{2n+1}, Tx_{2n+1})]^2, \frac{1}{2}[d(x_{2n}, Tx_{2n+1})]^2, \frac{1}{2}[d(x_{2n+1}, Sx_{2n})]^2\}}{d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})} \\ &= \frac{q \max \{[d(x_{2n}, x_{2n+1})]^2, [d(x_{2n+1}, x_{2n+2})]^2, \frac{1}{2}[d(x_{2n}, x_{2n+2})]^2\}}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}. \quad \ldots(3) \end{aligned}$$

If

$$\max \{[d(x_{2n}, x_{2n+1})]^2, [d(x_{2n+1}, x_{2n+2})]^2, \tfrac{1}{2}[d(x_{2n}, x_{2n+2})]^2\} = [d(x_{2n}, x_{2n+1})]^2 \quad \ldots(4)$$

then from (3) it follows that

$$[d(x_{2n+1}, x_{2n+2})]^2 + d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2}) - q[d(x_{2n}, x_{2n+1})]^2 \leqslant 0.$$

The positive root of the quadratic equation $t^2 + t - q = 0$ is

$$k = \tfrac{1}{2}[(1 + 4q)^{1/2} - 1]$$

and since $q < 1$, it follows that $k < 1$. Thus

$$d(x_{2n+1}, x_{2n+2}) \leqslant k d(x_{2n}, x_{2n+1}) \quad \ldots(5)$$

for $n = 0, 1, 2, \ldots$. If max of the three numbers in the braces on the left side in (4) is equal to $[d(x_{2n+1}, x_{2n+2})]^2$, then from (3) we have

$$d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) \leqslant q d(x_{2n+1}, x_{2n+2})$$

which is impossible since $q < 1$. If max of the three numbers in the braces on the left side in (4) is equal to $\frac{1}{2}[d(x_{2n}, x_{2n+2})]^2$, then from (3) we have

$$(2 - q)\,[d(x_{2n+1}, x_{2n+2})]^2 + (2 - 2q)\, d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2}) - q\,[d(x_{2n}, x_{2n+1})]^2 \leqslant 0.$$

The positive root of the quadratic equation $(2 - q)l^2 + (2 - 2q)l - q = 0$ is

$$r = q/(2 - q)$$

and since $q < 1$, it follows that $r < 1$. Thus

$$d(x_{2n+1}, x_{2n+2}) \leqslant r d(x_{2n}, x_{2n+1}) \qquad \text{...(6)}$$

for $n = 0, 1, 2, \ldots$. Hence by putting $\lambda = \max\{k, r\}$ we have from (5) and (6)

$$d(x_{2n+1}, x_{2n+2}) \leqslant \lambda d(x_{2n}, x_{2n+1}), \quad n = 0, 1, 2, \ldots \qquad \text{...(7)}$$

and so

$$d(x_{2n+1}, x_{2n+2}) \leqslant \lambda d(x_{2n}, x_{2n+1}) \leqslant \ldots \leqslant \lambda^{2n+1} d(x_0, x_1)$$

for $n = 0, 1, 2, \ldots$. Since $\lambda < 1$, it follows that the sequence defined in (2) is a Cauchy sequence in the complete metric space X and so has a limit z in X.

If we now suppose that $Sz \neq z$, then

$$\begin{aligned} d(z, Sz) &\leqslant d(z, x_{2n+2}) + d(x_{2n+2}, Sz) = d(z, x_{2n+2}) + d(Sz, Tx_{2n+1}) \\ &\leqslant d(z, x_{2n+2}) \\ &\quad + \frac{q \max\{[d(z, x_{2n+1})]^2, [d(z, Sz)]^2, [d(x_{2n+1}, x_{2n+2})]^2, \frac{1}{2}[d(z, x_{2n+2})]^2, \frac{1}{2}[d(x_{2n+1}, Sz)]^2\}}{d(z, Sz) + d(x_{2n+1}, x_{2n+2})} \end{aligned}$$

and on letting n tend to infinity we have

$$d(z, Sz) \leqslant q \max\{0, d(z, Sz), 0, 0, \tfrac{1}{2} d(z, Sz)\}$$

giving a contradiction since $q < 1$. It follows that z must be a fixed point of S. Similarly it follows that z is a fixed point of T. Thus z is a common fixed point of S and T.

Now suppose that $d(x, Sx) + d(y, Ty) = 0$ implies $d(Sx, Ty) = 0$ and that T has a second fixed point z'. Then $d(z, Sz) + d(z', Tz') = 0$ and so

$$d(z, z') = d(Sz, Tz') = 0.$$

It follows that $z = z'$ and so the common fixed point z of S and T in this case is unique. This completes the proof of the theorem.

We note that the conclusion of Theorem 1 remains valid if we replace the condition (1) by the condition of the type

$$d(Sx, Ty) \leqslant q \max\left\{\frac{[d(x, Sx)]^2 + [d(y, Ty)]^2}{d(x, Sx) + d(y, Ty)]}, \frac{1}{2}\frac{[d(x, Ty)]^2 + [d(y, Sx)]^2}{d(x, Sx) + d(y, Ty)}\right\} \qquad \text{...(1a)}$$

which in turn contains as a special case the mappings recently considered by Fisher (1977).

Iseki (1974) has proved a fixed point theorem which is a generalization of Maia's theorem (1968). We next establish the following variant of Iseki's result in the present setup of mappings.

Theorem 2 — Let X be a metric space with two metrices d and δ. If X satisfies the following conditions:

(i) $d(x, y) \leqslant \delta(x, y)$ for every x, y in X,

(ii) X is complete with respect to d,

(iii) two mappings $S, T: X \to X$ are continuous with respect to the metric d, and

$$\delta(Sx, Ty) \leqslant \frac{q \max\{[\delta(x,y)]^2, [\delta(x,Sx)]^2, [\delta(y,Ty)]^2, \frac{1}{2}[\delta(x,Ty)]^2, \frac{1}{2}[\delta(y,Sx)]^2\}}{\delta(x, Sx) + \delta(y, Ty)} \qquad \ldots(8)$$

for all x, y in X for which $\delta(x, Sx) + \delta(y, Ty) \neq 0$, $q \in (0, 1)$. Then S and T have a common fixed point z. Further, if $\delta(x, Sx) + \delta(y, Ty) = 0$ implies that $\delta(Sx, Ty) = 0$, then z is a unique fixed point of S and T.

PROOF : Let $x_0 \in X$ be arbitrary and define a sequence

$$x_{2n+1} = Sx_{2n}, \; x_{2n+2} = Tx_{2n+1}, \; n = 0, 1, 2, \ldots .$$

Then by following exactly the same steps as in the proof of Theorem 1 with suitable modifications we have

$$\delta(x_{2n+1}, x_{2n+2}) \leqslant \lambda^{2n+1}\, \delta(x_0, x_1)$$

where λ is as defined in the proof of Theorem 1. Now using $d \leqslant \delta$, we have

$$d(x_{2n+1}, x_{2n+2}) \leqslant \lambda^{2n+1}\, \delta(x_0, x_1).$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence with respect to d. Since X is complete with respect to d, the sequence $\{x_n\}$ has a limit z in X. Hence by the continuity of S with respect to the metric d,

$$z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = S \lim_{n\to\infty} x_{2n} = Sz.$$

Similarly we have $z = Tz$. Therefore z is a common fixed point of S and T. The uniqueness of z follows as in the proof of Theorem 1.

Ćirić (1974a) has established some interesting fixed point theorems for a family of generalized contractive type mappings. We next establish the following variants of Ćirić's results in the framework of the present setup of mappings.

Theorem 3 — Let T_0 and $\{T_n : n \in I^+\}$ (I^+ the positive integers) be mappings of a nonempty complete metric space X into itself satisfying the inequality

$$d(T_0x, T_ny) \leqslant \frac{q \max\{[d(x,y)]^2, [d(x,T_0x)]^2, [d(y,T_ny)]^2, \frac{1}{2}[d(x,T_ny)]^3, \frac{1}{2}[d(y,T_0x)]^2\}}{d(x,T_0x)+d(y,T_ny)} \quad \ldots(9)$$

for all x, y in X and for each $n = 1, 2, \ldots,$ for which $d(x, T_0x) + d(y, T_ny) \neq 0$, $q \in (0, 1)$, then there exists a fixed point $z \in X$ such that $T_nz = z$ for each $n = 0, 1, 2, \ldots$ and for arbitrary $x_0 \in X$ the sequence

$$x_0,\ x_1 = T_0x_0,\ x_2 = T_1x_1,\ x_3 = T_0x_2, \ldots,\ x_{2n-1} = T_0x_{2n-2},\ x_{2n} = T_nx_{2n-1}, \ldots$$

converging to z. Further, if $d(x, T_0x) + d(y, T_ny) = 0$ implies that $d(T_0x, T_ny) = 0$, then z is the unique fixed point of T_n for $n = 0, 1, 2, \ldots$.

Theorem 4 — Let $\mathcal{F} = \{T_\lambda : \lambda \in (\lambda)\}$ be a family of functions which maps a nonempty complete metric space X into itself. If there exists some $T_{\lambda_0} \in \mathcal{F}$ such that for each $T_\lambda \in \mathcal{F}$ $(\lambda \neq \lambda_0)$ there are positive integers i_λ and j_λ such that

$$d(T_{\lambda_0}^{i_\lambda}x, T_\lambda^{j_\lambda}y) \leqslant \frac{q \max\{[d(x,y)]^2, [d(x,T_{\lambda_0}^{i_\lambda}x)]^2, [d(y,T_\lambda^{j_\lambda}y)]^2, \frac{1}{2}[d(x,T_\lambda^{j_\lambda}y)]^2, \frac{1}{2}[d(y,T_{\lambda_0}^{i_\lambda}x)]^2\}}{d(x,T_{\lambda_0}^{i_\lambda}x)+d(y,T_\lambda^{j_\lambda}y)} \quad \ldots(10)$$

for all x, y in X for which $d(x, T_{\lambda_0}^{i_\lambda}x) + d(y, T_\lambda^{j_\lambda}y) \neq 0$, $q \in (0, 1)$, then every $T_\lambda \in \mathcal{F}$ has a fixed point z in X. Further, if $d(x, T_{\lambda_0}^{i_\lambda}x) + d(y, T_\lambda^{j_\lambda}y) = 0$ implies $d(T_{\lambda_0}^{i_\lambda}x, T_\lambda^{j_\lambda}y) = 0$, then z is the unique common fixed point for $\mathcal{F}$.

The details of the proofs of Theorems 3 and 4 follow by the similar arguments as in the proof of Theorem 1 in view of the proofs of Theorems 1 and 2 given by Ćirić (1974a) with suitable modifications, and we leave the details to the reader.

We finally prove a theorem analogous to Theorem 1 for compact metric spaces.

Theorem 5 — Let S and T be continuous mappings of a compact metric space X into itself such that

$$d(Sx, Ty) < \frac{\max\{[d(x,y)]^2, [d(x,Sx)]^2, [d(y,Ty)]^2, \frac{1}{2}[d(x,Ty)]^2, \frac{1}{2}[d(y,Sx)]^2\}}{d(x,Sx)+d(y,Ty)} \quad \ldots(11)$$

for all x, y in X for which $d(x, Sx) + d(y, Ty) \neq 0$. Then S and T have a common fixed point z. Further, if $d(x, Sx) + d(y, Ty) = 0$ implies $d(Sx, Ty) = 0$, then z is the unique fixed point of S and T.

PROOF: First of all suppose there exists $q < 1$ such that

$$d(Sx, Ty) \leqslant \frac{q \max \{[d(x, y)]^2, [d(x, Sx)]^2, [d(y, Ty)]^2, \frac{1}{2}[d(x, Ty)]^2, \frac{1}{2}[d(y, Sx)]^2\}}{d(x, Sx) + d(y, Ty)}$$

for all x, y in X. The result then follows from Theorem 1.

If no such q exists, then if $\{q_n\}$ is a monotonically increasing sequence of real numbers with $\lim_{n \to \infty} q_n = 1$, we can find sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$d(Sx_n, Ty_n) > \frac{q_n \max \{[d(x_n, y_n)]^2, [d(x_n, Sx_n)]^2, [d(y_n, Ty_n)]^2, \frac{1}{2}[d(x_n, Ty_n)]^2, \frac{1}{2}[d(y_n, Sx_n)]^2\}}{d(x_n, Sx_n) + d(y_n, Ty_n)}$$

for $n = 1, 2, \ldots$. Since X is compact, we can choose convergent subsequences $\{x_{n(r)}\} = \{x'_r\}$ and $\{y_{n(r)}\} = \{y'_r\}$ of $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively. Then if $\{q_{n(r)}\} = \{q'_r\}$, we have

$$d(Sx'_r, Ty'_r) > \frac{q'_r \max \{[d(x'_r, y'_r)]^2, [d(x'_r, Sx'_r)]^2, [d(y'_r, Ty'_r)]^2, \frac{1}{2}[d(x'_r, Ty'_r)]^2, \frac{1}{2}[d(y'_r, Sx'_r)]^2\}}{d(x'_r, Sx'_r) + d(y'_r, Ty'_r)}$$

for $r = 1, 2, \ldots$. Letting r tend to infinity we see that

$$d(Sx, Ty) \geqslant \frac{\max \{[d(x, y)]^2, [d(x, Sx)]^2, [d(y, Ty)]^2, \frac{1}{2}[d(x, Ty)]^2, \frac{1}{2}[d(y, Sx)]^2\}}{d(x, Sx) + d(y, Ty)}$$

This can happen only if

$$x = y = Sx = Ty = z \text{ (say)}$$

and this implies that z is a common fixed point of S and T. The uniqueness of z, if $d(x, Sx) + d(y, Ty) = 0$ implies that $d(Sx, Ty) = 0$, follows as in the proof of Theorem 2. This completes the proof of the theorem.

In concluding this paper, we note that the conclusions of Theorems 2–5 remain valid if we replace the conditions (8), (9), (10) and (11) by the corresponding modified version of the condition given in (1*a*).

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