Indian J. pure appl. Math., **10**(11) : 1369-1373, November 1979

THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

J. THANGAMANI

Ramanujan Institute, University of Madras, Madras 600005

(*Received* 16 *January* 1979; *after revision* 5 *May* 1979)

Let $F(z)$ be regular in the unit disc $|z| < 1$ and normalized by the conditions $F(0) = 0$ and $F'(0) = 1$. Let $2f(z) = (zF(z))'$. The paper deals with the mapping properties of $f(z)$ when $F(z)$ is known. If $F(z) = z + \sum_{2}^{\infty} a_n z^n$ is star-like (convex) with respect to symmetric points in $|z| < 1$ the region of star-likeness (convexity) with respect to symmetric points of $f(z)$ is determined. The results of this paper contain some results of Livingston (1966) as special case.

1. INTRODUCTION

Let S denote the class of functions $f(z)$ regular and univalent in the unit disc $|z| < 1$ with the normalization $f(0) = 0$ and $f'(0) = 1$. Let K, S^* and C denote the sub-classes of S consisting of convex, star-like and close-to-convex functions. Sakaguchi (1959) introduced the sub-class S_s^* of 'star-like functions with respect to symmetric points'. A necessary and sufficient condition for $f(z) = z + \sum_{2}^{\infty} a_n z^n$ regular in $|z| < 1$ to belong to S_s^* in $|z| < 1$ is that $\text{Re}\left(\frac{zf'(z)}{(f(z) - f(-z))}\right) > 0$, $|z| < 1$. The class S_s^* includes the class of convex functions and odd functions star-like with respect to the origin and is contained in the class of close-to-convex functions.

Recently, Das and Singh (1977) have introduced the sub-classes K_s consisting of 'convex functions with respect to symmetric points and C_s consisting of close-to-convex functions with respect to symmetric points'. A function $f(z) = z + \sum_{2}^{\infty} a_n z^n$ regular in $|z| < 1$ will belong to K_s iff $\text{Re}\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0$, $|z| > 1$ and $f(z)$ will belong to C_s if there exists a $g(z) \in S_s^*$ such that $\text{Re}\left(\frac{zf'(z)}{g(z) - g(-z)}\right) > 0$, $|z| < 1$. They have studied the behaviour of certain integral operators applied on the members of these classes and have established the following results:

"If $f(z)$ is a member of S_s^*, K_s or C_s then $F(z) = (2/z) \int_0^z f(t)\, dt$ is also a member of the same class respectively in $|z| < 1$." These results are analogous to the results of Libera (1965) for the class S^*, K and C.

The purpose of this paper is to study the converse problem, namely, if $F(z)$ belongs to S_s^* to find the radius r_0 such that $f(z) = \frac{1}{2}(zF(z))'$, belongs to S_s^* in $|z| < r_0$. The value of r_0 has been obtained when $F(z) \in S_s^*$, K_s and C_s respectively. The results obtained are analogous to the results of Livingston (1966) for the classes S^*, K and C. Similar problems have been studied by Bernardi (1970) and Padmanabhan (1969).

2. Theorems and their Proofs

Theorem 1 — If $F(z) \in S_s^*$ in $|z| < 1$ then $f(z) = \frac{1}{2}(zF(z))' \in S_s^*$ in $|z| < \frac{1}{2}$.

Proof : Since

$$F(z) \in S_s^*,\ \operatorname{Re}\left(\frac{zF'(z)}{F(z) - F(-z)}\right) > 0,\ |z| < 1 \qquad \ldots(1)$$

we have

$$\frac{2zF'(z)}{F(z) - F(-z)} = \frac{1 - w(z)}{1 + w(z)}\ [w(z) \text{ analytic in } |z| < 1,\ |w(z)| \leqslant 1,\ w(0) = 0] \qquad \ldots(2)$$

By our assumption we have $F(z) = (2/z) \int_0^z f(t)\, dt$ and hence

$$\frac{2zF'(z)}{F(z) - F(-z)} = \frac{2zF(z) - 2\int_0^z f(t)\, dt}{\int_0^z f(t)\, dt - \int_0^z f(-t)\, dt}. \qquad \ldots(3)$$

Using (2) and (3) by a straightforward computation we can show that

$$\frac{zf'(z)}{f(z) - f(-z)} = \frac{(2 + w(z) + w(-z))(1 - w(z))}{2(2 + w(z) + w(-z))(1 + w(z))} - \frac{2zw'(z)(1 + w(-z))}{2(2 + w(z) + w(-z))(1 + w(z))} \qquad \ldots(4)$$

$f \in S_s^*$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) = \frac{1}{2}\operatorname{Re}\left[\frac{1 - w(z)}{1 + w(z)} - \frac{2zw'(z)(1 + w(-z))}{(2 + w(z) + w(-z))(1 + w(z))}\right] > 0. \qquad \ldots(5)$$

Now $$\operatorname{Re}\left(\frac{1 - w(z)}{1 + w(z)}\right) = \frac{1 - |w(z)|^2}{|1 + w(z)|^2}$$

and

$$\operatorname{Re}\left(\frac{2zw'(z)(1 + w(-z))}{(2 + w(z) + w(-z))(1 + w(z))}\right) \leqslant \frac{2|z|\,|1 + w(-z)|}{|2 + w(z) + w(-z)|} \times \frac{(1 - |w(z)|^2)}{1 - |z|^2}$$

where we have used the well-known estimate $|w'(z)| \leqslant \dfrac{(1 - |w(z)|^2}{1 - |z|^2}$ for $|z| < 1$. Therefore (5) is true if

$$\frac{2|z|}{1 - |z|^2} < \frac{|2 + w(z) + w(-z)|}{|1 + w(z)|\,|1 + w(-z)|} = |h_1(z) + h_2(z)| \qquad \ldots(6)$$

where $h_1(z) = \dfrac{1}{1 + w(z)}$ and $h_2(z) = \dfrac{1}{1 + w(-z)}$. We have

$$1 - r \leqslant |1 + w(z)| \leqslant 1 + r$$

and therefore $\dfrac{1}{1 + r} \leqslant |h_i(z)| \leqslant \dfrac{1}{1 - r}$, $i = 1, 2$. Indeed $h_1(z)$ and $h_2(z)$ lie within the circle on the line joining $\dfrac{1}{1 + r}$ and $\dfrac{1}{1 - r}$ as diameter. This implies

$$|h_1(z) + h_2(z)| \geqslant \frac{2}{1 + r}$$

and hence from (6) it follows that $f(z) \in S_s^*$ if $\dfrac{2r}{1 - r^2} < \dfrac{2}{1 + r}$ or $r < \frac{1}{2}$.

Corollary — Let $F(z)$ be odd. Then $F(z) \in S_s^*$ implies that $F(z) \in S^*$ (starlike with respect to the origin). In this case $f(z) = \frac{1}{2}(zF(z))'$ is also odd and hence $f(z) \in S_s^*$ implies $f(z) \in S^*$. Also when $F(z)$ is odd the function $w(z)$ in (2) is even and substituting $w(z) = w(-z)$ in (6) we get

$$f \in S^* \text{ if } \frac{2|z|}{1 - |z|^2} < \frac{|2 + 2w(z)|}{|1 + w(z)|^2} = \frac{2}{|1 + w(z)|} \qquad \ldots(7)$$

and $$\frac{2}{|1 + w(z)|} \geqslant \frac{2}{1 + |z|}.$$

Putting $|z| = r$ we have $f \in S^*$ if $\dfrac{2r}{1 - r^2} < \dfrac{2}{1 + r}$, that is, $r < \frac{1}{2}$ which is due to Livingston (1966) for functions belonging to S^*.

Theorem 2 — If $F(z) \in K_s$ in $|z| < 1$ then $f(z) = \frac{1}{2}(zF(z))' \in K_s$ in $|z| < \frac{1}{2}$.

PROOF : $f(z) = \frac{(zF(z))'}{2} = \frac{zF'(z) + F(z)}{2}$. Hence $zf'(z) = \frac{(zh(z))'}{2}$

where $h(z) = zF'(z)$. Since $F(z) \in K_s$, $zF'(z) \in S_s^*$ (Das and Singh 1977). Therefore by Theorem 1, $zF'(z) \in S_s^*$ in $|z| < \frac{1}{2}$. Hence $f(z) \in K_s$ in $|z| < \frac{1}{2}$.

Theorem 3 — If $F(z) \in C_s$ in $|z| < 1$ with respect to the associated function $G(z) \in S_s^*$ then $f(z) = \frac{1}{2}(zF(z))' \in C_s$ with respect to $g(z) = \frac{1}{2}(zG(z))'$ in $|z| < \frac{1}{2}$.

PROOF : Since $G(z) \in S_s^*$ by Theorem 1,

$$g(z) = \tfrac{1}{2}(zG(z))' \in S_s^* \text{ in } |z| < \tfrac{1}{2} \qquad \ldots(8)$$

$$\text{Hence } \operatorname{Re}\left(\frac{zF'(z)}{G(z) - G(-z)}\right) > 0 \text{ in } |z| < 1. \qquad \ldots(9)$$

Putting $\frac{2zF'(z)}{G(z) - G(-z)} = p(z)$ where $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $|z| < 1$ we have

$$[2z^2F'(z)]' = p(z)\,[z(G(z) - G(-z))]' + p'(z)\,[z(G(z) - G(-z))]. \qquad \ldots(10)$$

$$\text{Hence} \qquad \frac{2zf'(z)}{g(z) - g(-z)} = \frac{2\,[z^2F'(z)]'}{[z(G(z) - G(-z))]'}$$

$$= p(z) + p'(z)\,\frac{z(G(z) - G(-z))}{[z(G(z) - G(-z))]'}. \qquad \ldots(11)$$

Let $\frac{1}{2}(G(z) - G(-z)) = K(z)$; since $G(z) \in S_s^*$, $K(z) \in S^*$ (Sakaguchi 1959). From (11) we have

$$\operatorname{Re}\left(\frac{2zf'(z)}{g(z) - g(-z)}\right) \geqslant \operatorname{Re} p(z) - |p'(z)|\left|\frac{zK(z)}{(zK(z))'}\right|. \qquad \ldots(12)$$

Applying the inequality $|p'(z)| \leqslant \frac{2 \operatorname{Re} p(z)}{1 - |z|^2}$ in (12) we get

$$\operatorname{Re}\left(\frac{2zf'(z)}{g(z) - g(-z)}\right) \geqslant \operatorname{Re} p(z)\left[1 - \frac{2}{1 - |z|^2}\cdot\left|\frac{zK(z)}{(zK(z))'}\right|\right]. \qquad \ldots(13)$$

Now putting $M(z) = \frac{(zK(z))'}{K(z)} = 1 + \frac{zK'(z)}{K(z)}$

we obtain $|M(z)| \geqslant \operatorname{Re} M(z) \geqslant 1 + \frac{1 - |z|}{1 + |z|} = \frac{2}{1 + |z|}$.

Hence $\left|\frac{z}{M(z)}\right| = \left|\frac{zK(z)}{(zK(z))'}\right| \leqslant \frac{|z| + |z|^2}{2}$.

From (13) putting $|z| = r$ we get

$$\text{Re}\left(\frac{2zf'(z)}{g(z) - g(-z)}\right) \geqslant \text{Re}\, p(z) \cdot \left[1 - \frac{2}{1 - r^2} \cdot \frac{(r + r^2)}{2}\right]$$

$$= \text{Re}\, p(z) \cdot \frac{(1 - 2r)}{(1 - r)}\Bigg]. \qquad \ldots(14)$$

Hence $\text{Re}\left(\dfrac{zf'(z)}{g(z) - g(-z)}\right) > 0$ if $r < \frac{1}{2}$.

Thus $f(z) \in C_s$ with respect to $g(z)$ in $|z| < \frac{1}{2}$.

Acknowledgement

The author is thankful to Prof. K. S. Padmanabhan for his kind encouragement and helpful guidance.

References

Bernardi, S. D. (1970). The radius of univalence of certain univalent functions. *Proc. Am. math., Soc.*, **24**, 312-18.

Das, R. N., and Singh, P. (1977). On sub-classes of Schlicht mapping. *Indian J. pure appl. Math.* **8**, 864-72.

Libera, R. J. (1965). Some classes of regular univalent functions. *Proc. Am. math. Soc.*, **16**, 755-58.

Livingston, A. E. (1966). On the radius of univalence of certain analytic functions. *Proc. Am. math. Soc.*, **17**, 352-57.

Nehari, Z. (1952). Conformal Mapping. McGraw-Hill Book Co., Inc., New York.

Padmanabhan, K. S. (1969). On the radius of univalence of certain classes of analytic functions. *J. Lond. math. Soc.* (2), **1**, 225-31.

Sakaguchi, K. (1959). On certain univalent mappings. *J. Math. Soc. Japan*, **11**, 72-80.