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RELATIVISTIC EQUATIONS FOR AXISYMMETRIC GRAVITATIONAL COLLAPSE WITH ESCAPING NEUTRINOS

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Einstein's field equations for the dynamics of a self-gravitating axially symmetric source of a perfect fluid, presented by Chandrasekhar and Friedman (1964), are modified to allow emission of neutrinos. The boundary conditions at the outer surface of the radiating axisymmetric source are obtained by matching to an exterior solution of an axisymmetric rotating, radiating core.

1. INTRODUCTION

Misner and Sharp (1964) presented the general relativity field equations for the dynamics of a self-gravitating sphere of an ideal fluid. As it is known that in a collapse of supernovae and quasi-stellar radio sources at extreme temperature attained in region of intense gravitational fields; neutrinos emission is possible. Hence Einstein's field equations for the collapse of a self-gravitating sphere of a perfect fluid were modified by Misner (1965) to allow an extremely simplified heat transfer process in which internal energy is converted, at some rate controlled by an equation of state, into an outward flux of neutrinos, which have no subsequent interaction with matter. When the neutrino flux vanishes, these modified equations reduce to those of Misner and Sharp (1964). The boundary conditions at the outer surface of these radiating sources are obtained by matching to an exterior Vaidya's (1951) radiating Schwarzschild metric (Misner *et al.* 1965).

Quasi-stellar radio sources are rotating objects. The main effect of rotation is to destroy radial character of a fluid. Due to roation a self gravitating collapsing source becomes axially symmetric. Here in this paper Einstein's field and fluid equations for the dynamics of a self-gravitating axially symmetric source of a perfect fluid, presented by Chandrasekhar and Friedman (1964), are modified to allow emission of neutrinos. An oblate spheroid of a perfect fluid subjected to gravitational and pressure gradient forces has been considered. As each element of the fluid will cool by emission of neutrinos at some rate depending on temperature and density, the fluid does not obey the simple adiabatic equation of state. Due to rotation of a collapsing source, the escaping neutrinos revolve around the source. Also it is assumed that they are neither scattered nor absorbed by the surrounding matter. The boundary conditions at the outer surface of a radiating oblate spheroidal object are obtained by matching to an axisymmetric radiating metric presented by the author (Patel 1978a).

2. The Stress-Energy Tensor

The stress-energy tensor for a perfect fluid is

$$T^{ij} = (\epsilon + p)\, u^i u^j + p g^{ij} \qquad ...(2.1)$$

where u^i is a fluid four-velocity vector and ϵ and p denote energy density and pressure respectively. The matter conservation is expressed by the equation of continuity

$$(Nu^i)_{;i} = 0 \qquad ...(2.2)$$

where N is baryon number density.

The stress-energy tensor fails to satisfy a local conservation law because of neutrinos emission. If $C(T, N)$ is a cooling rate of decrease in internal energy due to neutrinos emission per unit amount of matter, then

$$-NC = u^i(-\, T^j_{i;j}). \qquad ...(2.3)$$

Using eqns. (2.1) and (2.3), we get

$$-NC = (\epsilon u^i)_{;i} + p u^i_{;i} \cdot \qquad ...(2.4)$$

Again let us write

$$\epsilon = N(1 + e) \qquad ...(2.5)$$

where e is a specific internal energy that does not include rest-mass energy. Now use of eqns. (2.1) and (2.5), simplify eqn. (2.4) to the form

$$e_{,i} u^i = -\, C - \frac{P}{N}\, u^i_{;i} \qquad ...(2.6)$$

where comma and semi-colon signifies partial and covariant differentiation respectively, with respect to the index that follow. This is just the first law of thermodynamics with $-\, C$, as the heat input rate.

The neutrinos flux is described by the stress-energy tensor

$$E^{ij} = q k^i k^j, \; k^i k_i = 0 \qquad ...(2.7)$$

where q is flux energy density in some frame which depends on normalization of k^i. Now the local energy momentum conservation law is

$$(T^{ij} + E^{ij})_{;j} = 0. \qquad ...(2.8)$$

Use of eqn. (2.3) simplifies the above conservation law to the form

$$u^i(-E^j_{i;j}) = NC. \qquad \text{...(2.9)}$$

This equation represents the behaviour of the neutrinos flux.

3. Metric and Field Equations

To study the gravitational collapse of a rotating star with escaping neutrinos, suitable coordinates are the oblate spheroidal coordinates. As discussed by the author (Patel 1978b) axially symmetric space-time line element, in the oblate spheroidal coordinate system, is of the form

$$ds^2 = -e^{2\psi}dt^2 + e^{2\sigma}(d\phi - w\,dt)^2 + e^{2\beta}a^2(\theta^2 + \alpha^2)\left[\frac{d\theta^2}{1+\theta^2} + \frac{d\alpha^2}{1-\alpha^2}\right] \qquad \text{...(3.1)}$$

where ψ, σ, w and β are functions of t, θ and α and $0 \leqslant \theta < \infty$, $-1 \leqslant \alpha \leqslant 1$.

We shall make some general remarks on the notations that will be adopted here. The coordinates (x^0, x^1, x^2, x^3) and $(t, \phi, \theta, \alpha)$ will be used interchangeably. Latin letters as space-time indices are allowed the ranges 0, 1, 2 and 3; and Greek letters are restricted to 2 and 3 only. Summation over repeated tensor indices is assumed and restricted to their ranges. The convention of setting $c = G = 1$ is adopted.

To write out the field equations explicitly, we need expressions for the components of a four velocity vector u^i. In an axially symmetric case there can be no motion in the x^2- and x^3-directions; only rotational motion in the ϕ-direction specified by Ω can prevail. Hence with the definition

$$\frac{d\phi}{dt} = \Omega \quad \text{and} \quad V = (\Omega - w)\, e^{\sigma-\psi} \qquad \text{...(3.2)}$$

the components of the four-velocity vector are

$$u^0 = \frac{e^{-\psi}}{(1-V^2)^{1/2}};\quad u^1 = \frac{we^{-\psi} + Ve^{-\sigma}}{(1-V^2)^{1/2}};\quad u^2 = u^3 = 0 \qquad \text{...(3.3)}$$

$$u_0 = -\frac{(e^{\psi} + wVe^{\sigma})}{(1-V^2)^{1/2}};\quad u_1 = \frac{Ve^{\sigma}}{(1-V^2)^{1/2}};\quad u_2 = u_3 = 0. \qquad \text{...(3.4)}$$

As discussed by the author (Patel 1978a), the emitted neutrinos form an enveloping radiation zone surrounding a rotating axisymmetric collapsing star. Also according to our assumption the neutrinos are neither scattered nor absorbed by the surrounding matter. Hence for interior region also the null vector k^i takes the normalized form

$$k^i = (e^{-\psi},\ -e^{-\sigma} + we^{-\psi},\ 0,\ 0). \qquad \text{...(3.5)}$$

Now the hydrodynamic equation of motion follow from the total stress energy tensor satisfying local energy and momentum conservation laws

$$(T^{ij} + E^{ij})_{;j} = 0.$$

Using eqns. (2.1) and (2.3) in the above local energy and momentum conservation laws, we get the result of the form

$$(\epsilon + p)\, u^i_{;j}\, u^j = NCu^i - (g^{ij} + u^i u^j)\, p_{,j} - E^{ij}_{;j}. \qquad \text{...(3.6)}$$

t-component of the above equation is the hydrodynamic equation of motion. By using eqns. (3.3) and (3.5), we get equation of motion in the form

$$\frac{(\epsilon + p)\,\{VV_{,0} + V^2(1 - V^2)\,\sigma_{,0}\}}{(1 - V^2)^2} = \frac{NCe^{\psi}}{(1 - V^2)^{1/2}} - \frac{p_{,0}V^2}{1 - V^2} - q_{,0} - 2q(\sigma + \beta)_{,0}. \qquad \text{...(3.7)}$$

ϕ-component of eqn. (3.6) represents the equation of conservation of angular momentum per baryon. Hence the equation

$$(\epsilon + p)\left[\frac{VV_{,0}(we^{\sigma-\psi} + V)}{(1 - V^2)^2} + \frac{V_{,0} + V\sigma_{,0}(wVe^{(\sigma-\psi)} + 1)}{1 - V^2}\right]$$
$$= \frac{NC(we^{\sigma} + Ve^{\psi})}{(1 - V^2)^{1/2}} - \frac{p_{,0}V[1 + wVe^{(\sigma-\psi)}]}{1 - V^2}$$
$$+ [1 - we^{(\sigma-\psi)}]\,[q_{,0} + 2q(\sigma + \beta)_{,0}] \qquad \text{...(3.8)}$$

expresses the conservation of the angular momentum per baryon. The equations representing conservation of momentum conjugate to θ- and α-directions are

$$\frac{(\epsilon + p)}{(1 - V^2)}\,[\psi_{,\lambda} - V^2\sigma_{,\lambda} + Vw_{,\lambda}e^{(\sigma-\psi)}]$$
$$= -\,p_{,\lambda} - q\,[(\psi - \sigma)_{,\lambda} - w_{,\lambda}e^{(\sigma-\psi)}] \qquad \text{...(3.9)}$$

These equations look like hydrostatic balance of force. They help us in finding metric components on each t = constant, hyper-surface. A convenient boundary condition is to be selected at the outer surface, so that coordinate time becomes proper time there.

Now the eqns. (2.9) representing the behaviour of the neutrino flux reduces to

$$q_{,0} + 2q(\sigma + \beta)_{,0} = NC\left\{\frac{1 - V}{1 + V}\right\}^{1/2} e^{\psi}. \qquad \text{...(3.10)}$$

Then this result simplifies the equations of motion (3.7) and (3.8) to a simple form

$$\frac{(\epsilon + p)\,\{V_{,0} + V(1 - V^2)\,\sigma_{,0}\}}{1 - V^2} = NCe^{\psi}(1 - V^2)^{1/2} - Vp_{,0}. \qquad \text{...(3.11)}$$

Now the Einstein field equations for the interior region of an axially symmetric, rotating, collapsing source are

$$G^i_j = -8\pi(T^i_j + E^i_j). \qquad \text{...(3.12)}$$

Since

$$T^i_\lambda + E^i_\lambda = T^\lambda_i + E^\lambda_i = 0, \quad \text{for } i \neq \lambda \qquad \text{...(3.13)}$$

eqns. (3.12) imply that

$$G^i_\lambda = G^\lambda_i = 0, \quad \text{for } i \neq \lambda. \qquad \text{...(3.14)}$$

These equations give the following results:

$$w_{,0\lambda} + w_{,\lambda}(3\sigma - \psi)_{,0} = 0 \qquad \text{...(3.15)}$$

$$(\sigma + \beta)_{,0\lambda} + \sigma_{,\lambda}(\sigma - \beta)_{,0} - \psi_{,\lambda}(\sigma + \beta)_{,0} = 0 \qquad \text{...(3.16)}$$

$$(\sigma + \psi)_{,23} - \left(\beta_{,3} + \frac{\alpha}{\theta^2 + \alpha^2}\right)(\sigma + \psi)_{,2} - \left(\beta_{,2} + \frac{\theta}{\theta^2 + \alpha^2}\right)(\sigma + \psi)_{,3}$$
$$+ \psi_{,2}\psi_{,3} + \sigma_{,2}\sigma_{,3} - \tfrac{1}{2} w_{,2} w_{,3} \exp(2\sigma - 2\psi) = 0. \qquad \text{...(3.17)}$$

In writing the rest of Einstein's field equations it is convenient to introduce the following abbreviations.

$$\triangle w \equiv (1 + \theta^2) w_{,22} + (1 - \alpha^2) w_{,33} + \theta w_{,2} - \alpha w_{,3} \qquad \text{...(3.18)}$$

$$X \equiv (1 + \theta^2)^2 \{(\sigma + \psi)_{,22} + (\psi_{,2})^2 + (\sigma_{,2})^2\} \qquad \text{...(3.19)}$$

$$Y \equiv (1 - \alpha^2)^2 \{(\sigma + \psi)_{,33} + (\psi_{,3})^2 + (\sigma_{,3})^2\} \qquad \text{...(3.20)}$$

$$Z \equiv \triangle w + (1 + \theta^2) w_{,2}(3\sigma - \psi)_{,2} + (1 - \alpha^2) w_{,3}(3\sigma - \psi)_{,3} \qquad \text{...(3.21)}$$

$$G^0_0 = -8\pi(T^0_0 + E^0_0) \Rightarrow [\tfrac{1}{2}(\theta^2 - \alpha^2)\{\beta_{,00} + \beta_{,0}(\sigma - \psi + 2\beta)_{,0}\}$$
$$+ 2\beta_{,0}\sigma_{,0} + (\beta_{,0})^2] \exp(-2\psi)$$
$$+ [(\theta^2 + \alpha^2)\{(1 + \theta^2)(\sigma + \psi)_{,2}\beta_{,2} - (1 - \alpha^2)(\sigma + \psi)_{,3}\beta_{,3}\}$$
$$+ \triangle(\psi - \sigma) - (2 + \theta^2 - \alpha^2)\triangle\beta - X - Y]\frac{\exp(-2\beta)}{2a^2(\theta^2 + \alpha^2)}$$
$$- [2wZ + (1 - \theta^4)(w_{,2})^2 + (1 - \alpha^4)(w_{,3})^2]\frac{\exp(2\sigma - 2\psi - 2\beta)}{4a^2(\theta^2 + \alpha^2)}$$
$$= -8\pi\left[p - \frac{(p + \epsilon)\{1 + Vw\exp(\sigma - \psi)\}}{1 - V^2} - q\{1 - w\exp(\sigma - \psi)\}\right] \qquad \text{...(3.22)}$$

$$G_1^1 = -8\pi(T_1^1 + E_1^1) \Rightarrow [\tfrac{1}{2}(\theta^2 - \alpha^2)\{\beta_{,00} + \beta_{,0}(\sigma - \psi + 2\beta)_{,0}\}$$
$$+ 2\beta_{,00} - 2\beta_{,0}\psi_{,0} + 3(\beta_{,0})^2] \exp(-2\psi)$$
$$+ [\theta^2 + \alpha^2)\{(1 + \theta^2)(\sigma + \psi)_{,2}\beta_{,2} - (1 - \alpha^2)(\sigma + \psi)_{,3}\beta_{,3}\}$$
$$+ \triangle(\sigma - \psi) + 2(1 + \theta^2)(\sigma + \psi)_{,2}(\sigma - \psi)_{,2} - X - Y$$
$$+ 2(1 - \alpha^2)(\sigma + \psi)_{,3}(\sigma - \psi)_{,3} - (2 + \theta^2 - \alpha^2)\triangle\beta] \frac{\exp(-2\beta)}{2a^2(\theta^2 + \alpha^2)}$$
$$+ [2wZ + (1 + \theta^2)(3 + \theta^2)(w_{,2})^2 + (1 - \alpha^2)(3 - \alpha^2)(w_{,3})^2]$$
$$\times \frac{\exp 2(\sigma - \psi - \beta)}{4a^2(\theta^2 + \alpha^2)}$$
$$= -8\pi\left[\frac{p\{1 + Vw\exp(\sigma - \psi)\} + \epsilon V\{V + w\exp(\sigma - \psi)\}}{1 - V^2}\right.$$
$$\left. + q\{1 - w\exp(\sigma - \psi)\}\right] \qquad \ldots(3.23)$$

$$G_1^0 = -8\pi(T_1^0 + E_1^0) \Rightarrow Z\exp(\sigma - \psi - 2\beta)$$
$$= 16\pi a^2(\theta^2 + \alpha^2)\left[q - \frac{(\epsilon + p)V}{1 - V^2}\right] \qquad \ldots(3.24)$$

$$G_2^2 + G_3^3 = -8\pi[T_2^2 + T_3^3 + E_2^2 + E_3^3] \Rightarrow 2[\beta_{,00} + \beta_{,0}(\sigma - \psi - 2\beta)_{,0}$$
$$- (\beta_{,0})^2 + \sigma_{,00} - \psi_{,0}\sigma_{,0} + (\sigma_{,0})^2] \exp(-2\psi)$$
$$+ [-\triangle(\sigma + \psi) - 2(1 + \theta^2)(\sigma + \psi)_{,2}\psi_{,2}$$
$$- 2(1 - \alpha^2)(\sigma + \psi)_{,3}\psi_{,3}] \frac{\exp(-2\beta)}{a^2(\theta^2 + \alpha^2)}$$
$$= -16\pi p \qquad \ldots(3.25)$$

$$G_0^1 = -8\pi(T_0^1 + E_0^1) \Rightarrow 2w[-\beta_{,00} + \beta_{,0}(\sigma + \psi - \beta)_{,0}] \exp(-2\psi)$$
$$+ [-Z + 2w\{-\triangle(\sigma - \psi) + (1 + \theta^2)((\psi_{,2})^2 - (\sigma_{,2})^2)$$
$$+ (1 - \alpha^2)((\psi_{,3})^2 - (\sigma_{,3})^2)\}] \frac{\exp(-2\beta)}{2a^2(\theta^2 + \alpha^2)}$$
$$- [2w\{(1 + \theta^2)(w_{,2})^2 + (1 - \alpha^2)(w_{,3})^2\} + w^2 Z]$$
$$\times \frac{\exp 2(\sigma - \psi - \beta)}{2a^2(\theta^2 + \alpha^2)}$$
$$= -8\pi\left[-\frac{\epsilon + p}{1 - V^2}\{1 + Vw\exp(\sigma - \psi)\}\{V + w\exp(\sigma - \psi)\}\right.$$
$$\left. + q\{1 - w\exp(\sigma - \psi)\}^2\right] \exp(\psi - \sigma) \qquad \ldots(3.26)$$

$$G_2^2 - G_3^3 = 0 \Rightarrow -(\theta^2+\alpha^2)\,[\beta_{,00} + \beta_{,0}(\sigma - \psi + 2\beta)_{,0}]\exp(-2\psi)$$
$$+\Big[X - Y + (2+\theta^2-\alpha^2)\,\{(1-\alpha^2)\,\beta_{,3}(\sigma+\psi)_{,3}$$
$$-(1-\theta^2)\,\beta_{,2}(\sigma+\psi)_{,2}\} + (\theta^2+\alpha^2)\,\triangle\,\beta$$
$$+\frac{2(1+\theta^2)(1-\alpha^2)}{\theta^2+\alpha^2}\{\alpha(\sigma+\psi)_{,3} - \theta(\sigma+\psi)_{,2}\}\Big]\frac{\exp(-2\beta)}{a^2(\theta^2+\alpha^2)}$$
$$-[(1+\theta^2)^2(w_{,2})^2 - (1-\alpha^2)^2(w_{,3})^2]\frac{\exp 2(\sigma-\psi-\beta)}{2a^2(\theta^2+\alpha^2)}$$
$$= 0. \qquad \ldots(3.27)$$

A complete set of equations for an axially symmetric gravitational collapse with escaping neutrinos could consist of the eqns. (3.9), (3.17) and (3.24) giving the spatial derivatives of the independent variables σ, ψ, w, β and p on each t = constant hypersurface, together with eqns. (3.15), (3.16), (3.10) and (3.11) which give time derivatives of the variables q, V, (or Ω), p, w, σ, ψ and β, and Einstein's field eqns. (3.22), (3.23) and (3.25) to (3.27). Also equations of state (2.5) and (2.6) define p and ϵ as functions of N and e at desired temperature T. It is worth noting that if we consider $q = 0$, and all other variables are time independent, then these equations reduce to the field equations and equations of hydrodynamic equilibrium for axially symmetric stationary space-time, presented by Chandrasekhar and Friedman (1964).

4. Boundary Conditions

As neutrino flux permeates outside the region occupied by the rotating fluid spheroid $\theta = b$ = constant, the metric of the exterior space-time should be that of radiating axisymmetric metric presented by the author (Patel 1978a). Suitable form of this metric is

$$ds^2 = -2dt\,d\phi + (Bu + C)\,d\phi^2$$
$$+\,2a^2(\theta^2+\alpha^2)\,(M/R)^{1/2}\left[\frac{d\theta^2}{1+\theta^2} + \frac{d\alpha^2}{1-\alpha^2}\right] \qquad \ldots(4.1)$$

where

$$R = a\,(1+\theta^2-\alpha^2)^{1/2} \qquad \ldots(4.2a)$$

$$u = t - R \qquad \ldots(4.2b)$$

and M is total conserved mass energy associated with a radiating source. B and C are positive constants.

Now, according to the junction conditions of Lichnerowicz (1955), the metric tensor g_{ij} and their first derivatives $g_{ij,k}$ are continuous on the oblate spheroidal boundary

$$\theta = b = \text{constant.} \qquad \ldots(4.3)$$

Hence the interior metric (3.1) must match with the metric (4.1) on the oblate spheroidal boundary (4.3). Therefore

$$(\psi)_S = (-\sigma)_S \qquad \ldots(4.4a)$$

$$\{\exp(-2\psi)\}_S = Bu + C \qquad \ldots(4.4b)$$

$$(w)_S = \{\exp(2\psi)\}_S \qquad \ldots(4.4c)$$

$$(\beta)_S = [\tfrac{1}{4}\log(4M) - \tfrac{1}{8}\log(1 + \theta^2 - \alpha^2) - \tfrac{1}{4}\log a]_S \qquad \ldots(4.4d)$$

where S indicates that the values of the corresponding functions are to be taken on the boundary (4.3).

Now, it is easy to see from the eqn. (3.24) that the pressure p vanishes on the boundary (4.3). The eqn. (3.26) implies that matter density ϵ vanishes for $\theta \geqslant b$. Finally the eqn. (3.25) prescribes the flux of energy density q of escaping neutrinos at the boundary (4.3) by the formula

$$16\pi q = \left[\frac{B}{2(Bu + C)\sqrt{(MR)}}\right]_{\theta=b}. \qquad \ldots(4.5)$$

5. Conclusion

The field equations of a collaping axisymmetric rotating source with escaping neutrinos are obtained. As the collapsing core emits neutrinos, the surrounding region is an axially symmetric radiation zone. The metric tensor of the interior space matches with the radiating axisymmetric metric. Also the potential functions satisfy Lichnerowicz (1955) junction conditions of the surface of discontinuity.

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