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SOME BOUNDS FOR THE MULTIPLICATOR OF A FINITE GROUP

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In this paper, using cohomological methods, some bounds for the multiplicator of a finite group are obtained. The results obtained here improve upon and generalise some of the results obtained by Jones (1973, 1974).

1. INTRODUCTION

Let G be a finite group, K a normal subgroup of G and T the additive group of rationals mod 1 regarded as trivial G-module. Let $d(A)$ and $e(A)$ denote respectively the minimum number of generators and exponent of a group A. If G/K is cyclic, Jones [1974, Theorem 3.1 (i)] proved (using group-theoretic methods) that

$$| H^2(G, T) | \text{ divides } | H^2(K, T) | \; | K/K' | . \qquad \ldots(1.1)$$

In an earlier paper, Jones (1973, Theorem 4.1) also proved (again using group-theoretic methods) that if K is a central subgroup of G and $L = G/K$, then

(i) $| H^2(G, T) | \; | G' \cap K |$ divides $| H^2(K, T) | \; | H^2(L, T) | \; | K \otimes L |$

(ii) $d(H^2(G, T)) \leqslant d(H^2(K, T)) + d(H^2(L, T)) + d(K \otimes L)$

(iii) $e(H^2(G, T))$ divides $e(H^2(K, T)) \, . \, e(H^2(L, T)) \, . \, e(K \otimes L)$. ...(1.2)

Let IG denote the augmentation ideal of the integral group ring ZG. Using an extension of the exact sequence of Hochschild-Serre obtained by Vermani (1976), we prove the following.

Theorem 1 — Let G be a finite group and K a normal subgroup of G such that the inflation homomorphism inf: $H^2(G/K, T) \to H^2(G, T)$ is zero. Then

(i) $| H^2(G, T) |$ divides $| H^2(K, T) | \; | K/K' \otimes_L IL | \; | K' | \, / \, | [G, K] |$

(ii) $d(H^2(G, T)) \leqslant d(H^2(K, T)) + d(K/K' \otimes_L IL)$

(iii) $e(H^2(G, T))$ divides $e(H^2(K, T)) \, . \, e(K/K' \otimes_L IL)$

where $L = G/K$ and $[G, K]$ denotes the subgroup of G generated by elements

$$[x, k] = x^{-1}k^{-1}xk, \; x \in G, \; k \in K.$$

We deduce from Theorem 1 the following.

Corollary 1 — Let G be a finite group and K a normal subgroup of G such that G/K is cyclic. Then

$$| H^2(G, T) | \text{ divides } | H^2(K, T) | \; | K/[G, K] | .$$

Corollary 1 improves the bound (1.1) obtained by Jones. We give examples to show that (i) this bound is, in a sense, best possible and (ii) this gives a true improvement of the result of Jones. We also prove:

Theorem 2 — Let G be a finite group, K a normal subgroup of G and $L = G/K$. Then

(i) $| H^2(G, T) | \; | G' \cap K |$ divides
$$| H^2(K, T) | \; | H^2(L, T) | \; | K/K' \otimes_L IL | \; | K' |$$

(ii) $d(H^2(G, T)) \leqslant d(H^2(K, T)) + d(H^2(L, T)) + d(K/K' \otimes_L IL)$

(iii) $e(H^2(G, T))$ divides $e(H^2(K, T)) \,.\, e(H^2(L, T)) \,.\, e(K/K' \otimes_L IL)$.

We deduce from Theorem 2 the following.

Corollary 2 — Let G be a finite group and K a normal subgroup of G such that $K' = [G, K]$. Then

(i) $| H^2(G, T) | \; | G' \cap K |$ divides
$$| H^2(K, T) | \; | H^2(L, T) | \; | K/K' \otimes L/L' | \; | K' |$$

(ii) $d(H^2(G, T)) \leqslant d(H^2(K, T)) + d(H^2(L, T)) + d(K/K' \otimes L/L')$

(iii) $e(H^2(G, T))$ divides $e(H^2(K, T)) \,.\, e(H^2(L, T)) \,.\, e(K/K' \otimes L/L')$

where $L = G/K$.

Theorem 4.1 of Jones (1973) is then a particular case of Corollary 2.

2. Notations

If K is a normal subgroup of a group G, K/K' is regarded as a right G/K-module through conjugation by the elements of G. Then Hom $(K/K', T)$ is regarded as a left G/K-module under the diagonal action. Also for any left G-module B, Hom (IG, B) and Hom (ZG, B) are regarded as left G-modules under the diagonal action.

All other terms and notations are standard.

3. Proofs of Theorem 1 and Corollary 1

Proposition 1 — Let G be a finite group, K a normal subgroup of G and $L = G/K$. Then

(i) $| H^1(L, \text{Hom}\,(K/K', T)) | = | K/K' \otimes_L IL | \; | K' | \, / \, | [G, K] |$

(ii) $d(H^1(L, \text{Hom}(K/K', T))) \leqslant d(K/K' \otimes_L IL)$

(iii) $e(H^1(L, \text{Hom}(K/K', T)))$ divides $e(K/K' \otimes_L IL)$.

PROOF: Applying Hom (..., Hom $(K/K', T)$) to the ZL-free presentation

$$0 \to IL \to ZL \to Z \to 0$$

of Z, we get an exact sequence

$$0 \to \text{Hom}(K/K', T) \to \text{Hom}(ZL, \text{Hom}(K/K', T)) \to \text{Hom}(IL, \text{Hom}(K/K', T)) \to 0. \qquad \ldots(3.1)$$

$M = \text{Hom}(ZL, \text{Hom}(K/K'\ T))$ being a co-induced L-module, it follows from Proposition VI.11.4 of Hilton-Stammbach (1971) that $H^n(L, M) = 0$ for all $n \geqslant 1$. The long exact sequence of cohomology associated with (3.1) (Hilton-Stammbach 1971, p. 189) yields an exact sequence

$$0 \to \text{Hom}_L(K/K', T) \to \text{Hom}_L(ZL, \text{Hom}(K/K', T)) \to \text{Hom}_L(IL, \text{Hom}(K/K', T)) \to H^1(L, \text{Hom}(K/K', T)) \to 0.$$

Since

$$\text{Hom}_L(K/K', T) \cong \text{Hom}(K/[G, K], T) \qquad \text{(Cassels-Fröhlich 1967).}$$

$$\text{Hom}_L(ZL, \text{Hom}(K/K', T)) \cong \text{Hom}(K/K', T) \qquad \text{(Northcott 1966, Theorem 6, p. 24)}$$

$$\text{Hom}_L(IL, \text{Hom}(K/K', T)) \cong \text{Hom}(K/K' \otimes_L IL, T) \qquad \text{(Northcott 1966, 8.5.4, p. 166).}$$

and $\text{Hom}(A, T) \cong A$ for any finite Abelian group A (Hall 1959, Theorem 13.2.1, p. 195), the above exact sequence yields an exact sequence

$$0 \to K/[G, K] \to K/K' \to K/K' \otimes_L IL \to H^1(L, \text{Hom}(K/K', T)) \to 0.$$

The result then follows from this sequence.

Proof of Theorem 1

Let $H^2(G, T)^*$ be the subgroup of $H^2(G, T)$ defined by the exact sequence

$$0 \to H^2(G, T)^* \to H^2(G, T) \xrightarrow{\text{res}} H^2(K, T)$$

where res: $H^2(G, T) \to H^2(K, T)$ denotes the restriction homomorphism. Then

$$\left.\begin{aligned} &|H^2(G, T)| \text{ divides } |H^2(K, T)|\ |H^2(G, T)^*| \\ &d(H^2(G, T)) \leqslant d(H^2(K, T)) + d(H^2(G, T)^*) \\ &e(H^2(G, T)) \text{ divides } e(H^2(K, T)) \cdot e(H^2(G, T)^*). \end{aligned}\right\} \qquad \ldots(3.2)$$

Again the inflation homomorphism inf : $H^2(L, T) \to H^2(G, T)$ being zero, the initial part of the exact sequence of Theorem 3.9 of Vermani (1976) gives an exact sequence

$$0 \to H^2(G, T)^* \to H^1(L, \text{Hom}\,(K/K', T)). \qquad \ldots(3.3)$$

Now the theorem follows from (3.2), (3.3) and Proposition 1.

Proof of Corollary 1

Set $G/K = L$. Tensoring the exact sequence

$$0 \to IL \to ZL \to Z \to 0$$

with K/K' over Z, we get an exact sequence

$$0 \to K/K' \otimes IL \to K/K' \otimes ZL \to K/K' \to 0. \qquad \ldots(3.4)$$

Since $K/K' \otimes ZL$ is L-induced, $H_n(L, K/K' \otimes ZL) = 0$ for $n \geqslant 1$ (Hilton-Stammbach 1971, p. 210) and the long exact sequence in homology of groups associated with (3.4) (Hilton-Stammbach 1971, p. 189) gives rise to an exact sequence

$$0 \to H_1(L, K/K') \to K/K' \otimes_L IL \to K/K' \otimes_L ZL \to (K/K')_L \to 0.$$

As $$K/K' \otimes_L ZL \cong K/K' \text{ and } (K/K')_L \cong K/[G, K]$$

(Northcott 1966, p. 218; Hilton-Stammbach 1971, p. 203),

this sequence becomes

$$0 \to H_1(L, K/K') \to K/K' \otimes_L IL \to K/K' \to K/[G, K] \to 0.$$

Therefore

$$|\, H_1(L, K/K') \,|\ |\, K/K' \,| = |\, K/K' \otimes_L IL \,|\ |\, K/[G, K] \,|. \qquad \ldots(3.5)$$

Now L being cyclic, it follows from Proposition VI. 7.1 of Hilton-Stammbach (1971) that $|\, H_1(L, K/K') \,|$ divides $|\, K/[G, K] \,|$ and hence from (3.5),

$$|\, K/K' \otimes_L IL \,| \text{ divides } |\, K/K' \,|.$$

The result then follows from these remarks and Theorem 1.

4. Examples

Example 1 — Recall that if G denotes the group of order km generated by elements a, b with defining relations

$$a^k = 1,\ b^m = 1,\ bab^{-1} = a^r \text{ with } r^m \equiv 1 \pmod{k}$$

then G is the semi-direct product of the cyclic subgroups $A = \{a\}$ and $B = \{b\}$ and (Tahara 1972, Proposition 9, p. 377)

$$H^3(G, Z) \cong Z/(k, r-1, \sum_{i=0}^{m-1} r^i, (r^m - 1)/k)\, Z$$

where $(k, r - 1, \sum_{i=0}^{m-1} r^i, (r^m - 1)/k)$ denotes the greatest common divisor of k, $r - 1, \sum_{i=0}^{m-1} r^i$ and $(r^m - 1)/k$. Set $k = 4$, $m = 2$ and $r = 3$. Then

$$H^2(G, T) \cong H^3(G, Z) \cong Z/2Z.$$

Taking $K = A$, we see that G/K is cyclic; also $[G, K] = \{1, a^2\}$ so that we have,

$$| H^2(G, T) | = 2, | H^2(K, T) | = 1, | K/[G, K] | = 2.$$

Thus we find that the bound of Corollary 1 is attained showing thereby that the bound is, in a sense, the best possible.

Example 2 — Let P_1 and P_2 denote sylow 2-subgroups of the symmetric groups of degree 2 and 2^2 respectively. Then P_2 is the wreath product of P_1 and P_1 (Blackburn 1972). Let $G = P_2 = P_1 \wr P_1$. Then (Hall, 1959, p. 82) $G = \{(1), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$. Take $K = \{(1), (12), (34), (12)(34)\}$. K is a normal subgroup of G with G/K cyclic and $[G, K] = \{(1), (12)(34)\}$; therefore $| K/[G, K] | = 2$. Also $H^2(K, T) \cong H^2(Z_2 \oplus Z_2, T) \cong Z_2$ (Huppert 1967, p. 652) and we find that $| H^2(K, T) | \ | K/[G, K] | = 4 < | H^2(K, T) | \ | K/K' | = 8$. Thus we find that the bound of Corollary 1 is sharper as compared to the bound obtained by Jones [1974, Theorem 3.1 (i)].

5. Proofs of Theorem 2 and Corollary 2

Proof of Theorem 2

From the initial part of the exact sequence of Theorem 3.9 of Vermani (1976) namely,

$$0 \to \text{Hom}(G/G'K, T) \to \text{Hom}(G/G', T) \to \text{Hom}(K/[G, K], T) \to H^2(L, T) \to H^2(G, T)^* \to H^1(L, \text{Hom}(K/K', T))$$

it follows that

$$| H^2(G, T)^* | \text{ divides } | H^2(L, T) | \ | G/G' | \ | H^1(L, \text{Hom}(K/K', T)) | / | G/G'K | \ | K/[G, K] |$$

$$= | H^2(L, T) | \ | [G, K] | \ | H^1(L, \text{Hom}(K/K', T)) | / | G' \cap K |$$

$$d(H^2(G, T)^*) \leqslant d(H^2(L, T)) + d(H^1(L, \text{Hom}(K/K', T))).$$

and $\quad e(H^2(G, T)^*)$ divides $e(H^2(L, T)) \,.\, e(H^1(L, \text{Hom}(K/K', T)))$

The theorem then follows by combining these results with (3.2) and using Proposition 1.

Proof of Corollary 2

If $K' = [G, K]$ then K/K' is a trivial L-module. Therefore,

$$K/K' \otimes_L IL = K/K' \otimes L/L'$$

(Hilton-Stammbach 1971, p. 192). The corollary now follows from Theorem 2.

Remark : If K is a central subgroup of G then, $1 = K' = [G, K]$. Thus we see that Corollary 2 generalizes Theorem 4.1 of Jones (1973).

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