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PLANE SYMMETRIC SPACE-TIME OF CLASS-1 AND ELECTROMAGNETISM

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It is established that a class-1 plane symmetric space-time of Taub is inconsistent with the electromagnetic energy momentum distribution. A metric describing null electromagnetic fields of class, exceeding one has been obtained.

1. INTRODUCTION

The spherically symmetric space-time has played a very important role in the study of gravitational interpretation of equations of general relativity. Closely related with spherically symmetric metric is the plane-symmetric metric of Taub (1951). While studying the gravitational significance of class-1 spherically symmetric space-time Karmarkar (1948) obtained a necessary and sufficient condition for this space-time to be of class-1. Recently, we have investigated the conditions under which the plane-symmetric metric of Taub turns out to be class-1. We have found that, for the plane-symmetric metric

$$dS^2 = -A^2(x,t)\,dx^2 - S^2(x,t)\,(du^2 + u^2 d\phi^2) + C^2(x,t)\,dt^2 + 2D(x,t)\,dx\,dt \quad \ldots(1.1)$$

to be of class-1, Karmarkar's condition is necessary but not sufficient. It is easy to see that $S \equiv$ constant makes the space-time described by (1.1), of embedding class-1. If we set aside this trivial case, then the following theorem holds good (Pandey and Sharma 1977). "If $R_{2323} \neq 0$, then the plane symmetric line-element (1.1) will be of class-1 if and only if the Karmarkar's condition

$$R_{2323}R_{1414} - R_{1212}R_{3434} + R_{1224}R_{1334} = 0 \quad \ldots(1.2)$$

is satisfied.

If $R_{2323} = 0$ and $S \neq$ constant, Karmarkar's conditions are satisfied and still the space-time described by (1.1) is not of class-1."

In this paper, we have investigated the question of existence of plane-symmetric electromagnetic fields of class-1 in the light of the theorem stated above. It is established that the plane-symmetric space-time (1.1) cannot describe an electromagnetic field if it is of class-1. It is well known that spherically symmetric electromagnetic fields of class-1 do not exist (Pandey and Kansal 1968).

2. Some Geometrical Considerations

The surfaces $S(x, t)$ = constant, are space-like, null or time-like according as

$$g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} < \; = \text{ or } > 0 \qquad \ldots(2.1)$$

The condition (2.1) is equivalent to

$$R_{2323} > \; = \text{ or } < 0 \qquad \ldots(2.2)$$

as it can be seen from (1.1), that

$$R_{2323} = \frac{S^2u^2}{A^2C^2 + D^2}(C^2S_1^2 - 2DS_1S_4 - A^2S_4^2). \qquad \ldots(2.3)$$

We investigate, in what follows the forms of the metric that are possible in these three cases separately.

Case I — When $S_1 \neq 0$ and $(C^2S_1^2 - 2DS_1S_4 - A^2S_4^2) > 0$, we can make use of the transformation

$$\bar{x} = S(x, t) \qquad \ldots(2.4)$$

bringing (1.1) to the form

$$ds^2 = -\frac{A^2}{S_1^2}d\bar{x}^2 - \bar{x}^2(du^2 + u^2d\phi^2) + \left(C^2 - \frac{A^2S_4^2}{S_1^2} - \frac{2DS_4}{S_1}\right)dt^2 + 2\left(\frac{D}{S_1} + \frac{A^2S_4}{S_1^2}\right)d\bar{x}\,dt. \qquad \ldots(2.5)$$

By a further transformation for t, given by

$$d\tau = \eta\left[\left(\frac{D}{S_1} + \frac{A^2S_4}{S_1^2}\right)d\bar{x} + \left(C^2 - \frac{A^2S_4^2}{S_1^2} - \frac{2DS_4}{S_1}\right)dt\right] \qquad \ldots(2.6)$$

where $\eta = \eta(\bar{x}, t)$ is an integrating factor, which exists since

$$C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 \neq 0$$

the metric (2.5) goes into the orthogonal form

$$ds^2 = -\left[\frac{A^2}{S_1^2} + \left(\frac{D}{S_1} + \frac{S_4A^2}{S_1^2}\right)^2\left(C^2 - \frac{A^2S_4^2}{S_1^2} - \frac{2DS_4}{S_1}\right)^{-1}\right]d\bar{x}^2 - \bar{x}^2(du^2 + u^2d\phi^2) + \left(C^2 - \frac{A^2S_4^2}{S_1^2} - \frac{2DS_4}{S_1}\right)^{-1}\frac{d\tau^2}{\eta^2}. \qquad \ldots(2.7)$$

This metric can be expressed as

$$ds^2 = -A^2dx^2 - x^2(du^2 + u^2d\phi^2) + C^2dt^2 \qquad \ldots(2.8)$$

where $A = A(x, t),\ C = C(x, t)$.

Case II — When $S_4 \neq 0$ and $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 < 0$, we can similarly make use of the transformation

$$\bar{t} = S(x, t) \qquad \text{...(2.9)}$$

which brings (1.1) to the form

$$ds^2 = -\left(A^2 + \frac{2DS_1}{S_4} - \frac{C^2S_1^2}{S_4^2}\right) dx^2 - \bar{t}^2(du^2 + u^2d\phi^2) + \frac{C^2}{S_4^2} d\bar{t}^2 + 2\left(\frac{D}{S_4} - \frac{C^2S_1}{S_4^2}\right) dx\, d\bar{t}. \qquad \text{...(2.10)}$$

Further, applying the transformation

$$d\bar{x} = \eta(x, \bar{t})\left[\left(-A^2 - \frac{2DS_1}{S_4} + \frac{C^2S_1^2}{S_4^2}\right) dx + \left(\frac{D}{S_4} - \frac{C^2S_1}{S_4^2}\right) d\bar{t}\right] \qquad \text{...(2.11)}$$

where $\eta(x, \bar{t})$ is again an integrating factor which exists since

$$C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 \neq 0$$

we see that (2.10) again goes into orthogonal form

$$ds^2 = \frac{1}{\eta^2}\left(-A^2 - \frac{2DS_1}{S_4} + \frac{C^2S_1^2}{S_4^2}\right)^{-1} d\bar{x}^2 - \bar{t}^2(du^2 + u^2d\phi^2) + \left[\frac{C^2}{S_4^2} - \left(\frac{D}{S_4} - \frac{C^2S_1}{S_4^2}\right)^2\left(-A^2 - \frac{2DS_1}{S_4} + \frac{C^2S_1^2}{S_4^2}\right)\right]^{-1} d\bar{t}^2. \qquad \text{...(2.12)}$$

This form of the metric (2.12) can be expressed as

$$ds^2 = -A^2d\bar{x}^2 - t^2(du^2 + u^2d\phi^2) + C^2dt^2 \qquad \text{...(2.13)}$$

where $A = A(x, t),\ C = C(x, t)$.

Case III — When $S_1 \neq 0$ and $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 = 0$, we can make use of the transformation (2.4) to get (2.5) which together with the condition

$$C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 = 0$$

yields the form of the metric as

$$ds^2 = -\frac{A^2}{S_1^2} d\bar{x}^2 - \bar{x}^2(du^2 + u^2d\phi^2) + 2\left(\frac{D}{S_1} + \frac{S_4A^2}{S_1^2}\right) d\bar{x}\, dt. \qquad \text{...(2.14)}$$

This form of the metric (2.14) can be expressed as

$$ds^2 = -A^2dx^2 - x^2(du^2 + u^2d\phi^2) + 2D\, dx\, dt. \qquad \text{...(2.15)}$$

where $A = A(x, t);\ D = D(x, t)$.

Case IV — When $S_4 \neq 0$ and $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 = 0$, we can again make use of the transformation (2.9) to get the form of the metric as (2.10) which in view of the condition

$$C^2S_1^2 - 2DS_1S - A^2S_4^2 = 0$$

takes the form

$$ds^2 = -\bar{t}^2(du^2 + u^2d\phi^2) + \frac{C^2}{S_4^2}\,d\bar{t}^2 + 2\left(\frac{D}{S_4} - \frac{C^2S_1}{S_4^2}\right)dx\,d\bar{t}. \quad ...(2.16)$$

This form of the metric (2.16) can be expressed as

$$ds^2 = -t^2(du^2 + u^2d\phi^2) + C^2dt^2 + 2D\,dx\,dt \quad ...(2.17)$$

where $C = C(x, t)$; $D = D(x, t)$.

Case V — When $S_1 = 0$ and $S_4 = 0$ i.e. $S \equiv$ constant $= K$ (say) and consequently $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 = 0$, the line element (1.1) takes the form

$$ds^2 = -A^2dx^2 - K^2(du^2 + u^2d\phi^2) + C^2dt^2 + 2D\,dx\,dt \quad ...(2.18)$$

In the first two cases, $R_{2323} \neq 0$ and in the last three cases $R_{2323} = 0$. So in the cases I and II, Karmarkar's condition (1.2) is necessary and sufficient for the metric (2.8) and (2.13) to be of class-1 while in case III, it has already been proved (Pandey and Sharma 1977) that the metric (2.15) cannot be of class-1. Similar is the case IV. In case V, the line element (2.18) is always of class-1 if it is non-flat. By the transformation

$$Z^1 = Su\cos\phi;\ Z^2 = Su\sin\phi;\ Z^3 = \tfrac{1}{2}S(u^2 + 1);\ Z^4 = \tfrac{1}{2}S(u^2 - 1)$$

the line element (1.1) reduces to

$$ds^2 = -(dZ^1)^2 - (dZ^2)^2 + (dZ^3)^2 - (dZ^4)^2 + d\Sigma^2$$

where $d\Sigma^2 = -A^2dx^2 + 2D\,dx\,dt + C^2dt^2$, which is 2-space imbedable in a 3-dimensional flat space. Thus it is established that (1.1) can, in general, be imbedded in a flat-space of seven dimensions and hence its class cannot exceed 3.

3. Impossibility of Plane Symmetric Electromagnetic Field of Class One

In this section it is established that the electromagnetic energy momentum tensor

$$T_{ij} = \tfrac{1}{4}g_{ij}F_{mn}F^{mn} - F_{im}F_j^{m} \quad ...(3.1)$$

as a source-term in Einstein's field equations

$$-8\pi T_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R \quad ...(3.2)$$

is not compatible with plane symmetric line-element of class-1. Here the skew-symmetric tensor F_{ij} is the electromagnetic field tensor.

In Einstein-Maxwell theory of electromagnetism, the field tensor F_{ij} appears as a source of gravitational significance through (3.1) and (3.2). A set of necessary as well as sufficient condition for the existence of F_{ij}, giving rise to an energy momentum tensor of the form (3.1) are supplied by Rainich algebraic relations (Witten 1962)

$$T = 0;\ T^m_i\, T^j_m = \sigma\delta^j_i\ ;\ \sigma = \tfrac{1}{4}\, T_{mn}T^{mn};\ \sigma \geqslant 0 \qquad \text{...(3.3)}$$

In consequence of (3.3) and the fact that T^1_1, $T^2_2 = T^3_3$, T^4_4, T^1_4 and T^4_1 are the only no-nzero components of the energy momentum tensor T^j_i for a general plane-symmetric space-time, we find that Rainich conditions assume the form

$$\left.\begin{aligned} &T^1_1 + T^2_2 + T^3_3 + T^4_4 = 0,\ (T^1_1)^2 + T^4_1\, T^1_4 = \sigma \\ &(T^1_1)^2 - (T^4_4)^2 = 0,\ (T^2_2)^2 = (T^3_3)^2 = \sigma \\ &(T^1_1 + T^4_4)\, T^4_1 = 0,\ (T^1_1 + T^4_4)\, T^1_4 = 0. \end{aligned}\right\} \qquad \text{...(3.4)}$$

From the third equation of the last set, two distinct cases arise, namely

$$\text{(a)}\quad T^1_1 + T^4_4 = 0, \qquad \text{(b)}\quad T^1_1 - T^4_4 = 0. \qquad \text{...(3.5)}$$

In case (a), it follows from (3.4), that

$$T^2_2 = T^3_3 = 0,\ \sigma = 0,\ T^1_1\, T^4_4 - T^4_1\, T^1_4 = 0.$$

Since $\sigma = 0$, the case (a) would give null electromagnetic fields and eqns. (3.4) reduce to

$$T^2_2 = 0,\ T^1_1 + T^4_4 = 0,\ T^1_1\, T^4_4 - T^4_1\, T^1_4 = 0. \qquad \text{...(3.6)}$$

In case (b), it is clear from the first of (3.4) that $T^2_2 = T^3_3 \neq 0$ if $T^1_1 = T^4_4 \neq 0$. Consequently $\sigma \neq 0$ and the case (b), therefore, would give non-null electromagnetic fields and the condition (3.4) reduce to

$$T^1_4 = 0,\ T^4_1 = 0,\ T^1_1 - T^4_4 = 0,\ T^1_1 + T^2_2 = 0. \qquad \text{...(3.7)}$$

Now we see that electromagnetic energy momentum tensor given by (3.1) is not compatible with plane-symmetric line-element of class-1 in each case mentioned in section 2.

Case I: (When $S_1 \neq 0$ and $(C^2S_1^2 - 2DS_1S_4 - A^2S_4^2) > 0$) — In this case the form of the metric is given by (2.8) and the field eqns. (3.2) yield

$$\left.\begin{aligned}
8\pi T^1_1 &= -\frac{(2C_1x + C)}{Cx^2A^2} \\
8\pi T^2_2 &= \frac{1}{A^2}\left[\frac{A_1}{Ax} + \frac{AA_{44}}{C^2} - \frac{C_{11}}{C} - \frac{C_1}{Cx} + \frac{C_1A_1}{AC} - \frac{AA_4C_4}{C^3}\right] \\
8\pi T^2_2 &= 8\pi T^3_3,\ 8\pi T^4_4 = \frac{2A_1}{A^3x} - \frac{1}{A^2x^2} \\
8\pi T^1_4 &= -\frac{2A_4}{xA^3},\ 8\pi T^4_1 = \frac{2A_4}{xAC^2}.
\end{aligned}\right\} \quad \ldots(3.8)$$

The remaining field equations are identically satisfied.

Now first we deal with the case (3.6), giving rise to null-electromagnetic fields. In consequence of (3.6), Karmarkar's necessary and sufficient conditions (1.2) for the plane symmetric Taub's metric to be of class-1 which in terms of T^j_i can be expressed as

$$3F^2 + 8\pi F(4T^2_2 - T^1_1 - T^4_4) - 64\pi^2(T^1_1\, T^4_4 - T^4_1\, T^1_4) = 0 \quad \ldots(3.9)$$

where $F = -\, R_{2323}/u^2S^4$,

yields

$$1/x^2A^2 = 0 \quad \ldots(3.10)$$

which is not consistent. Therefore, null electromagnetic field of class-1 is not compatible with the metric (2.8).

If the electromagnetic field is non-null, (3.7) holds good. In view of (3.8) eqns. (3.7) yield

$$\left.\begin{aligned}
&\frac{2A_4}{xA^3} = 0,\ \frac{2A_4}{xC^2A} = 0,\ \frac{C_1}{C} + \frac{A_1}{A} = 0 \\
&-\frac{3C_1}{Cx} - \frac{1}{x^2} + \frac{A_1}{Ax} - \frac{C_{11}}{C} + \frac{A_1C_1}{AC} = 0.
\end{aligned}\right\} \quad \ldots(3.11)$$

In consequence of (3.7), we see that class-1 condition (3.9) reduces to

$$\frac{1}{x^2} + \frac{4C_1}{xC} + \frac{C_1^2}{C^2} = 0. \quad \ldots(3.12)$$

Solution of (3.12) and first three equations of (3.11) is

$$A = K_2x^{-\alpha},\ C = K_1(t)\, x^{\alpha}$$

where $\alpha = -\,2 \pm \sqrt{3}$.

On substituting this solution in the last of (3.11), we see that it is not satisfied. Therefore, we conclude that eqns. (3.11) and (3.12) are not consistent. Therefore, non-null electromagnetic field of class-1 is also not possible in the case of the metric (2.8).

Case II (When $S_4 \neq 0$ and $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 < 0$) — Proceeding exactly on the same lines it can be seen that electromagnetic field of class-1 is not possible in this case also.

Case III and Case IV (When $S_1 \neq 0$ and $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 = 0$; $S_4 \neq 0$ and $C^2S_1^2 - 2DS_1S_4 - A^2S_4^2 = 0$) — In these cases it has been mentioned earlier in section 2 that the plane symmetric line element cannot be of class-1 which evidently proves the impossibility of electromagnetic fields of class-1 in the plane symmetric space-time described by the Taub's metric.

Case V (When $S \equiv$ constant) — In this case the first equation of Rainich relations (3.3) demands the flatness of the space-time described by (2.18). So the space-time described by (2.18) cannot give the electromagnetic field of class-1. It has also been verified that this metric is also inconsistent with a perfect fluid distribution. Being of class-1, it cannot describe matter free gravitational field. It, therefore, follows that this metric is devoid of any physical meaning.

We have thus established that the plane-symmetric space-time of class-1 is not compatible with an electromagnetic energy distribution. In this respect the behaviour of the plane-symmetric space-time is analogous to the spherically symmetric space-time (Pandey and Kansal 1968).

4. A Plane Symmetric Electromagnetic Field of Class Two or More

In the previous section we have shown that class-1 plane-symmetric space-time is not compatible with the electromagnetic energy momentum tensor. Hence, plane symmetric electromagnetic field, if they exist, will be of class 2 or more. We will show here that the plane symmetric metric (2.15) which is not of class-1 describes a null electromagnetic field in a special case.

The non-zero components of T_i^j for the metric (2.15) are given by

$$\left.\begin{aligned} 8\pi T_2^2 &= 8\pi T_3^3 = \frac{1}{D^2}\left[D_{14} + A_4^2 + AA_{44} - \frac{D_4}{D}(D_1 + AA_4)\right] \\ 8\pi T_1^4 &= \frac{2}{xD^2}(D_1 + AA_4). \end{aligned}\right\} \quad \ldots(4.1)$$

The scalar $T = T_i^i$ is given by

$$8\pi T = \frac{2}{D^2}\left[D_{14} + A_4^2 + AA_{44} - \frac{D_4}{D}(D_1 + AA_4)\right]. \quad \ldots(4.2)$$

For an electomagnetic field $T = 0$. In view of (4.2), the vanishing of the scalar T leads to the solution

$$D_1 + AA_4 = D \, . \, F(x) \qquad \ldots(4.3)$$

where $F(x)$ is arbitrary.

In consequence of (4.1) and (4.3), the only surviving component of T_i^j is

$$8\pi T_1^4 = 2F(x)/xD \qquad \ldots(4.4)$$

On account of (4.3), the Rainich algebraic conditions (3.6) are satisfied identically. Therefore, we have found the plane-symmetric metric (2.15) together with (4.3), representing a null electromagnetic field whose class can be at the most three. It is worthwhile mentioning here that the space-time (2.15) is of class-2 in the special case when $A = A(x)$ and $D = D(x)$. It can be verified that the transformation

$$\left.\begin{aligned} Z^1 &= -\tfrac{1}{2}\int A^2\,dx + \frac{x}{2}(u^2 - 1) \\ Z^2 &= -\tfrac{1}{2}\int A^2\,dx + \tfrac{1}{2}x(u^2 + 1) \\ Z^3 &= xu\cos\phi, \; Z^4 = xu\sin\phi \\ Z^5 &= \frac{1}{\sqrt{2}}\left(\int D\,dx - t\right), \; Z^6 = \frac{1}{\sqrt{2}}\left(\int D\,dx + t\right) \end{aligned}\right\} \qquad \ldots(4.5)$$

send (2.15), with A and D as functions of x only, into

$$ds^2 = -(dZ^1)^2 + (dZ^2)^2 - (dZ^3)^2 - (dZ^4)^2 - (dZ^5)^2 + (dZ^6)^2.$$

From what we have done above, it is clear that the space-time described by eqns. (4.5) represents a null electromagnetic field of class 2.

5. Conclusion

It is shown that the plane symmetric space-time of Taub (1951), is in general, of class-3. In case it is of class-1, it cannot accommodate an electromagnetic energy momentum distribution. All electromagnetic fields, which are plane-symmetric in the sense of Taub (1951) are either of class-2 or class-3.

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