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FLUCTUATING FLOW BETWEEN TWO COAXIAL CYLINDERS

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Flow and heat transfer of an incompressible viscous fluid between two concentric cylinders when the inner cylinder is at rest and the velocity of the outer one fluctuates about a steady mean have been studied. Equations of motion and energy have been directly integrated. Low and high frequency considerations have been made. Skin friction and Nusselt number at both the walls have been calculated. The study is, however, restricted to viscous fluids with Prandtl number equal to unity.

INTRODUCTION

Khamuri (1957) has discussed the flow between two porous circular cylinders due to the oscillations of the inner cylinder. Pulsating flow due to pressure gradient in the direction of the flow has been considered by Uchida (1956), Fan and Chao (1965) and Rama Charyulu (1966). Fluctuating flow of a fluid between two concentric cylinders could be entirely in response to the fluctuations in the pressure gradient alone or in the velocity of either the inner or outer cylinder or in both. The flow generated in such a geometry due to periodic axial pressure gradient alone fluctuating in magnitude about a steady mean has been considered by Nayak (1974). The problem is effectively an extension of the one considered by Uchida (1956).

We consider here the flow of a viscous fluid generated in response to fluctuations in the axial velocity of the outer cylinder. Closed form solutions for the velocity and temperature distribution have been obtained. Analysis has also been extended to low and high frequency approximations. We observe that the rate of flow of a fluid through an annular pipe could be increased by increasing the velocity fluctuations of the outer pipe. This result could be profitably employed in practice where pumping out or pumping in of a fluid is involved.

FORMULATION OF THE PROBLEM

Two infinitely long coaxial right circular cylinders of inner radius b and outer radius a are considered. It is assumed that the space between them is filled by a viscous incompressible fluid and that the outer cylinder moves in the axial direction

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alone with a velocity $W = W_0(1 + \epsilon e^{i\omega' t'})$ where W_0, ϵ, ω' are constants. Further the cylinders are assumed to be at the constant temperatures T_a (outer) and T_b(inner). We wish to study the response in the forced convectional flow and temperature of the fluid due to the fluctuations in the axial velocity of the outer cylinder. It is also assumed that the centre of the cylindrical co-ordinates is taken at the centre of the coaxial cylinders. If u, v, W are the velocity components of the fluid in the radial, azimuthal and axial directions and T, the temperature, then the statement of the problem can be made as follows:

$$\left.\begin{array}{l} u = v = 0 \quad \text{at} \quad r' = a, b \\ W = 0 \quad \text{at} \quad r' = b \quad \text{and} \quad W = W_0(1 + \epsilon e^{i\omega' t'}) \quad \text{at} \quad r' = a \\ T = T_a \quad \text{at} \quad r' = a \quad \text{and} \quad T = T_b \quad \text{at} \quad r' = b. \end{array}\right\} \quad \ldots(1)$$

As the cylinders are infinitely long and have axial symmetry, the velocity and the temperature functions are considered as independent of z and θ and as such are functions of r' and t' only.

Thus $u = u(r', t')$, $v = v(r', t')$, $W = W(r', t')$ and $T = T(r', t')$. Further since there is no force to generate any acceleration in the radial and azimuthal directions, we assume $u(r', t') = v(r', t') = 0$ in the entire fluid region and as such the problem reduces to one of finding $W(r', t')$ and $T(r', t')$.

Basic Equations

The equations of motion and energy for an incompressible fluid with constant properties in cylindrical polar co-ordinates are

$$\frac{\partial W}{\partial t'} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 W}{\partial r'^2} + \frac{1}{r'}\frac{\partial W}{\partial r'}\right) \quad \ldots(2)$$

$$\rho C_p \frac{\partial T}{\partial t'} = k\left(\frac{\partial^2 T}{\partial r'^2} + \frac{1}{r'}\frac{\partial T}{\partial r'}\right) + \mu\left(\frac{\partial W}{\partial r'}\right)^2 \quad \ldots(3)$$

where ρ is the density, C_p the specific heat, μ the coefficient of viscosity and k the thermal conductivity of the fluid.

The above two equations are to be solved subject to the boundary conditions (1). We write

$$\left.\begin{array}{l} W = F_1(r') + \epsilon F_2(r')\, e^{i\omega' t'} \\ T - T_b = T_1(r') + \epsilon T_2(r')\, e^{i\omega' t'} \end{array}\right\} \quad \ldots(4)$$

as solutions where the second term in each of (4) is the fluctuating part associated with an arbitrary parameter ϵ of small magnitude. Since the r-directional equation of momentum is $-(1/\rho)(\partial p/\partial r') = 0$ and since there is axial symmetry in the problem, p and $-(1/\rho)(\partial p/\partial z)$ can be functions of z and t. But since the other terms in (2) do not depend upon z, $\partial p/\partial z$ from the condition that eqn. (2) is also valid at the walls, in particular, at the outer wall where $W = W_0(1 + \epsilon e^{i\omega' t'})$, will be a function of t alone.

Thus $$-\frac{1}{\rho}\frac{\partial p}{\partial z} = i\,.\,\epsilon\omega' e^{i\omega' t'}\,.\,W_0.$$

Introducing (4) into (3) and (2), we obtain

$$\epsilon F_2 e^{i\omega' t'} = i\epsilon\omega'\,.\,e^{i\omega' t'}\,.\,W_0 + \frac{\mu}{\rho}(F_1'' + \epsilon F_2'' e^{i\omega' t'}) + \frac{1}{r'}(F_1' + \epsilon F_2' e^{i\omega' t'}) \quad \ldots(5)$$

and $$\rho C_p\, i\,.\,\omega'\,.\,T_2 e^{i\omega' t'} = k(T_1'' + \epsilon T_2'' e^{i\omega' t'}) + \frac{1}{r'}(T_1' + \epsilon T_2' e^{i\omega' t'}) + \mu\{F_1'^2 + 2\epsilon F_1' F_2' e^{i\omega' t'} + O(\epsilon^2)\} \quad \ldots(6)$$

where dashes mean differentiation with respect to r'.

The boundary conditions to be satisfied are

$$\left.\begin{aligned} &r' = b,\ F_1 = 0 = F_2,\ T_1 = 0 = T_2 \\ &r' = a,\ F_1 = W_0 = F_2,\ T_1 = T_a - T_b,\ T_2 = 0. \end{aligned}\right\} \quad \ldots(7)$$

Introducing dimensionless quantities

$$W_0 f_1(\eta) = F_1(r'),\ W_0 f_2(\eta) = F_2(r')$$

with $$\eta = \frac{r'}{a},\ \omega = \frac{\omega' a}{W_0},\ t = \frac{t' W_0^2}{\nu} \text{ and } (T_a - T_b)\,\theta_1 = T_1;\ (T_a - T_b)\,\theta_2 = T_2$$

where ν is the kinematic viscosity, into (5) and (6) we obtain

$$i\epsilon e^{i\omega t}(1 - f_2) + L\left(f_1'' + \frac{1}{\eta}f_1'\right) + L\left(f_2'' + \frac{1}{\eta}f_2'\right)e^{i\omega t} = 0 \quad \ldots(8)$$

and $$i\epsilon\theta_2 e^{i\omega t} = Pr\,.\,L(\theta_1'' + \theta_2'' e^{i\omega t}) + \frac{1}{\eta}(\theta_1' + \theta_2' e^{i\omega t}) + M(f_1'^2 + 2\epsilon f_1' f_2' e^{i\omega t}) \quad \ldots(9)$$

where $$L = \frac{\mu}{\omega a\rho W_0},\ M = \frac{W_0}{a\rho C_p(T_a - T_b)\,\omega},\ Pr = \frac{k}{\mu C_p} \quad \text{(Prandtl number).}$$

Terms of the order ϵ^2 and higher orders are neglected and dashes mean differentiation with respect to η. We restrict our studies to fluids whose Prandtl number is equal to 1.

Equating the terms independent of ϵ and harmonic terms from (8) separately to zero, we have

$$\left.\begin{aligned} & i(1-f_2) + L\left(f_2'' + \frac{1}{\eta} f_2'\right) = 0 \\ & L\left(f_1'' + \frac{1}{\eta} f_1'\right) = 0. \end{aligned}\right\} \quad \ldots(10)$$

The relevant boundary conditions are

$$\left.\begin{aligned} & \eta = \sigma;\ f_1 = f_2 = 0 \\ & \eta = 1;\ f_1 = f_2 = 1, \quad \text{where} \quad \sigma = b/a. \end{aligned}\right\} \quad \ldots(11)$$

On similar treatment (9) yields

$$\left.\begin{aligned} & L\left(\theta_1'' + \frac{1}{\eta}\theta_1'\right) + Mf_1'^2 = 0 \\ & i\theta_2 = L\left(\theta_2'' + \frac{1}{\eta}\theta_2'\right) + 2Mf_1' f_2' \end{aligned}\right\} \quad \ldots(12)$$

with boundary conditions

$$\left.\begin{aligned} & \eta = \sigma;\ \theta_1 = \theta_2 = 0 \\ & \eta = 1;\ \theta_1 = 1,\ \theta_2 = 0. \end{aligned}\right\} \quad \ldots(13)$$

Solutions of the Equations

The eqns. (10) and (12) can be exactly solved. If we write $f_2 = f_{2r} + if_{2i}$ and $\theta_2 = \theta_{2r} + i\theta_{2i}$, the solutions are

$$f_1 = \frac{\log(\sigma/\eta)}{\log \sigma} \quad \ldots(14)$$

$$\left.\begin{aligned} & f_{2r} = 1 + H_1 \, Ber\, \lambda\eta - H_2 \, Bei\, \lambda\eta + H_3 \, Ker\, \lambda\eta - H_4 \, Kei\, \lambda\eta \\ & f_{2i} = H_2 \, Ber\, \lambda\eta + H_1 \, Bei\, \lambda\eta + H_3 \, Kei\, \lambda\eta + H_4 \, Ker\, \lambda\eta \end{aligned}\right\} \quad \ldots(15)$$

where $$H_1 = -\frac{A\, Ker\, \lambda + B\, Kei\, \lambda}{A^2 + B^2}, \quad H_2 = \frac{B\, Ker\, \lambda - A\, Kei\, \lambda}{A^2 + B^2}$$

$$H_3 = \frac{A\, Ber\, \lambda + B\, Bei\, \lambda}{A^2 + B^2}, \quad H_4 = \frac{A\, Bei\, \lambda - B\, Ber\, \lambda}{A^2 + B^2}$$

$$A = (Ker\, \lambda\, Ber\, \lambda\sigma - Kei\, \lambda\, Bei\, \lambda\sigma) - (Ker\, \lambda\sigma\, Ber\, \lambda - Kei\, \lambda\sigma\, Bei\, \lambda)$$

$$B = (Kei\, \lambda\, Ber\, \lambda\sigma + Ker\, \lambda\, Bei\, \lambda\sigma) - (Kei\, \lambda\sigma\, Ber\, \lambda + Ker\, \lambda\sigma\, Bei\, \lambda)$$

and $$\lambda^2 = \frac{1}{L}$$

$$\theta_1 = -\frac{\lambda_1}{2}(\log \eta)^2 + \frac{\lambda_1 (\log \sigma)^2 \log \eta - 2 \log \eta + 1}{\log \sigma} \qquad \ldots(16)$$

where $\lambda_1 = \dfrac{M}{L(\log \sigma)^2}$

$$\theta_{2r} = \varphi_1(\eta) + \varphi_3(\eta) + \varphi_5(\eta)$$

$$\theta_{2i} = \varphi_2(\eta) + \varphi_4(\eta) + \varphi_6(\eta)$$

where
$$\varphi_1 = \frac{M_1 \log \sigma}{A^2 + B^2}\{A(Ker\,\lambda\,Ber\,\lambda\eta - Kei\,\lambda\,Bei\,\lambda\eta) + B(Kei\,\lambda\,Ber\,\lambda\eta + Ker\,\lambda\,Bei\,\lambda\eta)\}$$

$$\varphi_2 = \frac{M_1 \log \sigma}{A^2 + B^2}\{A(Kei\,\lambda\,Ber\,\lambda\eta + Ker\,\lambda\,Bei\,\lambda\eta) - B(Ker\,\lambda\,Ber\,\lambda\eta - Kei\,\lambda\,Bei\,\lambda\eta)\}$$

$$\varphi_3 = -\frac{M_1 \log \sigma}{A^2 + B^2}\{A(Ber\,\lambda\,Ker\,\lambda\eta - Bei\,\lambda\,Kei\,\lambda\eta) + B(Bei\,\lambda\,Ker\,\lambda\eta + Ber\,\lambda\,Kei\,\lambda\eta)\}$$

$$\phi_4 = \frac{M_1 \log \sigma}{A^2 + B^2}\{A(Bei\,\lambda\,Ker\,\lambda\eta + Ber\,\lambda\,Kei\,\lambda\eta) - B(Ber\,\lambda\,Ker\,\lambda\eta - Bei\,\lambda\,Kei\,\lambda\eta)\}$$

$$\varphi_5 = \frac{M_1 \log \eta}{A^2 + B^2}\{A(Ber\,\lambda\,Ker\,\lambda\eta + Kei\,\lambda\,Bei\,\lambda\eta - Ker\,\lambda\,Ber\,\lambda\eta - Bei\,\lambda\,Kei\,\lambda\eta) + B(Ber\,\lambda\,Kei\,\lambda\eta + Bei\,\lambda\,Kei\,\lambda\eta - Ker\,\lambda\,Bei\,\lambda\eta - Kei\,\lambda\,Ber\,\lambda\eta)\}$$

$$\varphi_6 = \frac{M_1 \log \eta}{A^2 + B^2}\{A(Ber\,\lambda\,Kei\,\lambda\eta + Bei\,\lambda\,Ker\,\lambda\eta - Ker\,\lambda\,Bei\,\lambda\eta - Kei\,\lambda\,Ber\,\lambda\eta) - B(Ker\,\lambda\eta\,Ber\,\lambda + Kei\,\lambda\,Bei\,\lambda\eta - Ker\,\lambda\,Ber\,\lambda\eta - Bei\,\lambda\,Kei\,\lambda\eta)\}$$

$$M_1 = 2M/L \log \sigma$$

Skin Friction

The response of the skin frictions and Nusselt numbers at the walls to the fluctuations in the velocity and temperature of the walls are of immense practical interest and have been studied in the normal (intermediate case) and in the extreme cases namely those of low frequency and of large frequency.

$$(\text{Skin friction})\ {}^{\text{Outer}}_{\text{Inner}}\ \text{cylinder} = \mu W_0 \left[- \frac{1}{\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix} \log \sigma\right)} + \lambda\epsilon \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} \cos\left(\omega t + \begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix}\right)\right] \quad \ldots(17)$$

with
$$\begin{vmatrix} z_1 \\ z_2 \end{vmatrix} = \left[f'^2_{2r}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right) + f'^2_{2i}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right)\right]^{1/2}$$

and
$$\begin{smallmatrix}\alpha_1\\ \alpha_2\end{smallmatrix} = \tan^{-1} \frac{f'_{2i}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right)}{f'_{2r}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right)}$$

where the upper quantity is to be chosen in case of the outer and the lower in case of the inner cylinder. With the same convention

$$(\text{Nusselt number})\ {}^{\text{Outer}}_{\text{Inner}}\ \text{cylinder} = \left[\frac{\frac{\lambda_1}{2}\ (\log \sigma)^2 - 1}{\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right) \log \sigma} + \epsilon \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} \cos\left(\omega t + \begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)\right] \quad \ldots(18)$$

where
$$\begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \left[\theta'^2_{2r}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right) + \theta'^2_{2i}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right)\right]^{1/2}$$

and
$$\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix} = \tan^{-1} \frac{\theta'_{2i}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right)}{\theta'_{2r}\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right)}$$

Low Frequency

(i) $(\text{Skin friction})\ {}^{\text{Outer}}_{\text{Inner}}\ \text{cylinder} = \mu W_0 \begin{vmatrix} z_3 \\ z_4 \end{vmatrix} \left\{ -1 + \epsilon \cos\left(\omega t + \begin{smallmatrix}\alpha_3\\ \alpha_4\end{smallmatrix}\right)\right\}$

when $|z_3| = (\log \sigma)^{-1}$; $|z_4| = (\sigma \log \sigma)^{-1}$; $\alpha_3 = \alpha_4 = \tan^{-1} \theta$

(ii) $$(\text{Nusselt number})\ {}^{\text{Outer}}_{\text{Inner}}\ \text{cylinder} = \left\{ \left(\frac{\begin{smallmatrix}+1\\ -1\end{smallmatrix}\ \lambda_1}{2}\right) (\log \sigma)^2 - 1 \right\} \times \frac{1}{\left(\begin{smallmatrix}1\\ \sigma\end{smallmatrix}\right) \log \sigma} - \epsilon \begin{vmatrix} y_3 \\ y_4 \end{vmatrix} \cos\left(\omega t + \begin{smallmatrix}\beta_3\\ \beta_4\end{smallmatrix}\right)$$

when $|y_3| = M/\sigma$; $|y_4| = M\lambda$; $\beta_3 = \beta_4 = \tan^{-1} \theta$

Large Frequency

For large frequency let us denote the amplitudes and the phase angles at the walls in case of the skin friction by $z_5, z_6, \alpha_5, \alpha_6$ and those in case of the Nusselt number by $y_5, y_6, \beta_5, \beta_6$.

The corresponding expressions for these are obtained on approximating the asymptotic expressions of *Ber*, *Bei*, *Ker*, *Kei* functions in (17) and (18) for skin friction and Nusselt number.

Discussion

Velocity of the fluid seems to increase with ϵ except for values of ωt near $\pi/2$. As ωt increases with the passage of time there is a slowing down of the flow, though for large values of λ (say $\lambda \geqslant 10$), the rule breaks. For $\lambda = 10$, we observe the velocity profile attains a peak which approaches the outer moving wall with increasing time. For small values of ($\lambda = 5$), the velocity and temperature profiles are linear. (Figs. 1, 2, 3). $\left[\text{In the graphs } \eta \text{ is written for } \eta' = \frac{\eta - \sigma}{1 - \sigma}.\right]$

For higher values of $\lambda(\lambda \geqslant 10)$ we observe that the fluid velocity records a steep rise with increasing distance from the stationary inner wall attaining a maximum at a certain distance and then falling to a minimum value near the moving outer wall. For $\omega t = \pi/2$ the maximum velocity attained is about 40% higher than that of the moving wall while for $\omega t = 5\pi/3$ it is 60% higher. For $\omega t = 5\pi/3$ the fluid

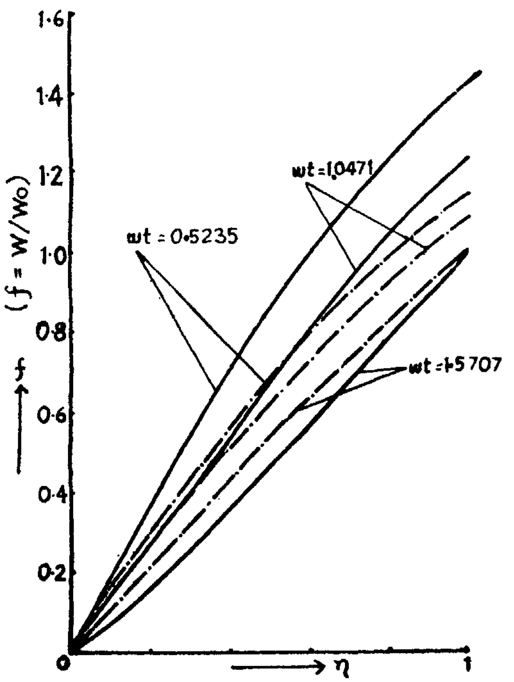

Fig. 1. Velocity profile ($\lambda = 5$, $\sigma = .5$). —— $\varepsilon = 0.5$, —·— $\varepsilon = 0.2$.

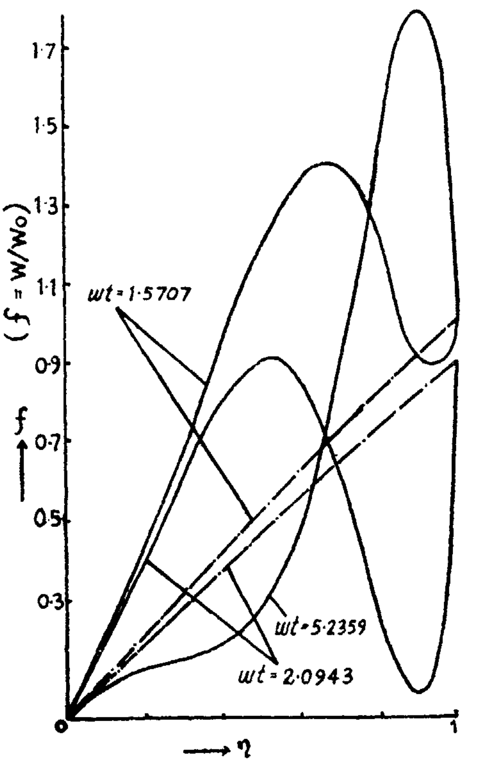

Fig. 2. Velocity profile. —— $\lambda = 10$, $\varepsilon = .02$, $\sigma = .5$; —·— $\lambda = 5$, $\varepsilon = .2$, $\sigma = .7$.

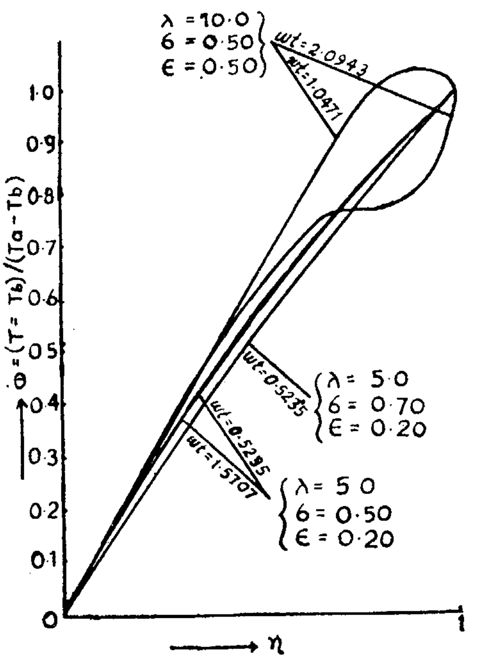

FIG. 3. Temperature profile.

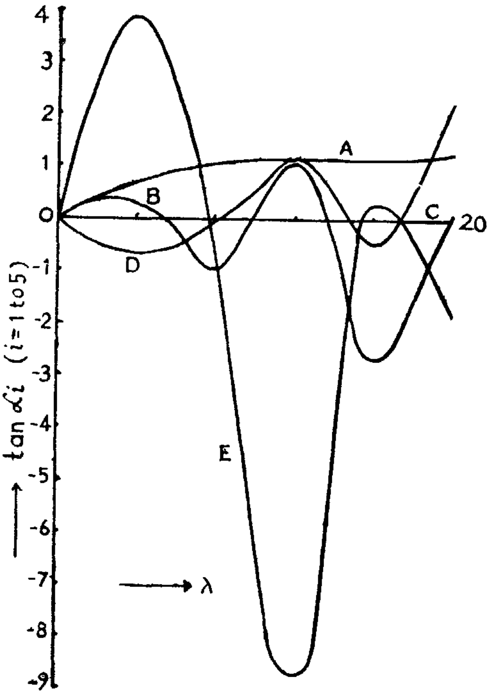

FIG. 4. Skin friction phase. A = Intermediate ($\eta = 0$), B = Intermediate ($\eta = 1$) C = Small ($\eta = 0$), D = Large ($\eta = 0$), E = Large ($\eta = 1$).

layers $0.75 < \eta < 1$ move faster than the outer wall. For $\omega t = 2\pi/3$ the maximum is equal to the velocity of the moving wall and is attained very near the midstream. It is interesting to see that a fluid layer near the moving wall ($\eta \doteq 0.9$) is almost dead slow (Fig. 2).

As expected fluctuations of small frequency in the velocity of the outer cylinder does not create any response in the phase of either skin friction or the Nusselt number at the walls.

For intermediate frequencies, skin friction at the inner wall takes a phase lead which maintains a steady value about 1 for $\lambda \geqslant 10$, but Nusselt number at the inner wall fluctuates in unision. For such frequencies phase of the skin friction and the Nusselt number at the outer wall takes a lead and a lag alternately as the frequency parameter λ increases from 0 starting with a phase lead in case of the former and with a phase lag in the case of the latter. For large frequencies, the phase lead frequency profiles of the skin friction and the Nusselt number are exactly similar, showing that wall oscillate in perfect unision. Both those have the

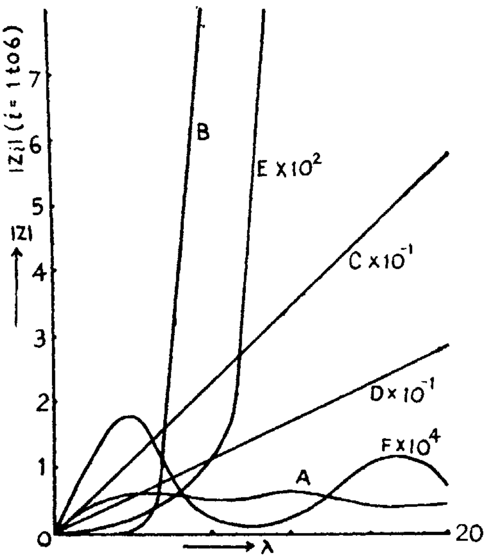

FIG. 5. Skin friction amplitude. A = Intermediate ($\eta = 0$), B = Intermediate ($\eta = 1$), C = Small ($\eta = 0$), D = Small ($\eta = 1$), E = Large ($\eta = 0$), F = Large ($\eta = 1$).

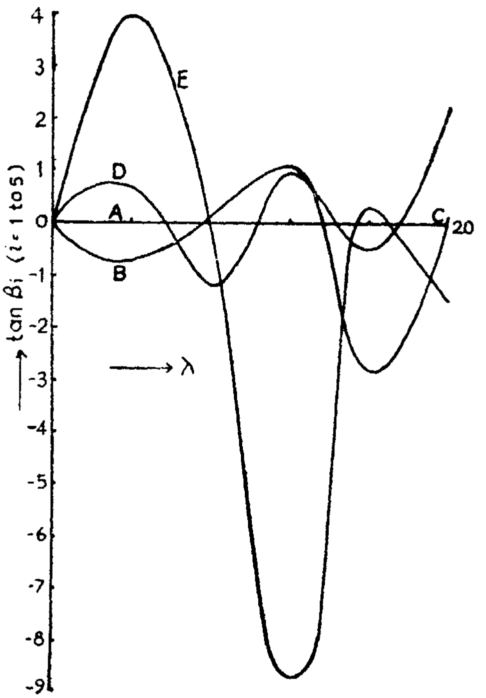

FIG. 6. Nusselt number phase. A = Intermediate ($\eta = 0$), B = Intermediate ($\eta = 1$), C = Low, D = Large ($\eta = 0$), E = Large ($\eta = 1$).

maximum phase lead for $\lambda = 4$ and the maximum phase lag for $\lambda = 12$. (Figs. 4 and 6);

For small frequencies the amplitude of the fluctuating skin friction at both the walls increase linearly with an increase in λ, the values at the inner wall being larger. For intermediate frequencies it attains almost a steady value at the inner wall but oscillations are already of large amplitudes when $\lambda = 8$. For large frequencies, oscillations at the inner wall are larger than those at the outer wall (Fig. 5). For small frequencies Nusselt number at the inner wall has amplitudes of constant magnitude ($|z| = 2$) but at the outer wall it increases linearly with λ. For intermediate frequencies Nusselt number at the linner wall rises from 0 to the same vlaue $|z| = 2$ as for small frequencies, but at the outer wall it has comparatively large amplitudes for values of $\lambda \geqslant 10$. For large frequencies Nusselt number has very small amplitudes at the outer wall, but very large ones at the inner ones for values of $\lambda > 16$. For $\lambda = 8$, Nusselt number at both the walls is of negligible amplitude (Fig. 7).

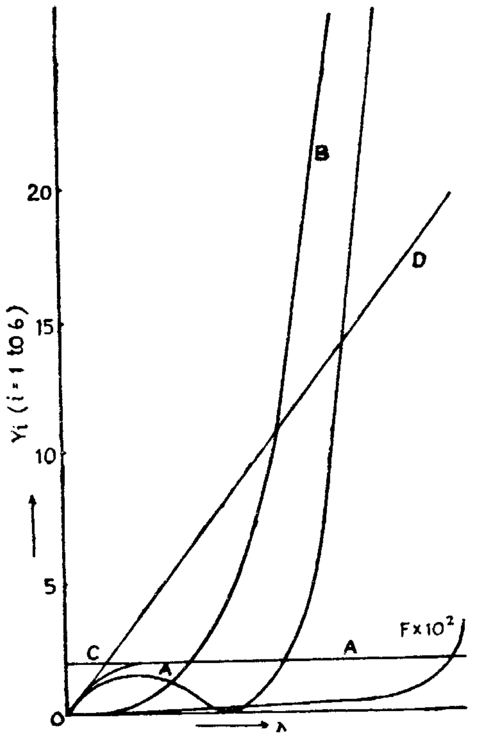

FIG. 7. Nusselt number amplitude. A = Intermediate ($\eta = 0$), B= Intermediate ($\eta = 1$), C = Small ($\eta = 0$), D = Small ($\eta = 1$), E = Large ($\eta = 0$), F = Large ($\eta = 1$).

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