Indian J. pure appl. Math., **10**(11) : 1422-1431, November 1979

SOME RESULTS ON k-REGULAR SEMIGROUPS

K. S. HARINATH

Department of Mathematics, Bangalore University, Bangalore 560001

(*Received* 25 *October* 1978)

A few results applicable to the class of generalized regular semigroups called k-regular semigroups are presented in this paper. Such classes of semigroups arise naturally in endomorphism rings of certain nice classes of abelian groups. Results on idempotent-separating congruences and classifications of right k-inverse semigroups are also included. Furthermore, a few new results on right inverse semigroups has been added. The theory when restricted to finite semigroups has immense applications to psychology, game theory, etc., via the mathematical theory of complexity.

In this paper, we present a few results applicable to the class of generalized regular semigroups called k-regular semigroups in accordance with the terminology used in the treatise by Fuchs (1973, cf. p. 239). This generalized concept for rings is presented in detail in the above treatise and such classes of rings (or semigroups) arise or occur naturally as endomorphism rings of certain nice classes of abelian groups and rings. In the following, we extend the results on k-regular rings and also on regular semigroups to include k-regular semigroups.

We have five sections in this expository paper containing the results. In section 1 we introduce the definitions together with examples and elementary results by recalling the requisite preliminaries. In section 2 we present a few elementary results on k-regular semigroups. Section 3 gives a few results on idempotent-separating congruences. Section 4 classifies right k-inverse semigroups. Section 5 sets forth a few new results on right inverse semigroups. Many results which are established in the theory of regular semigroups are extended in the obvious manner to include k-regular semigroups. For the sake of brevity, the proofs are indicated and may be constructed in parallel to the analogous results available in the classical theory. On the whole, this paper abstractly extends the theory of regular semigroups to a wider class of semigroups. The theory when restricted to finite semigroups has important applications in several branches of science such as psychology, physics, game theory, etc., via the mathematical theory of complexity due to Rhodes (1971).

1. INTRODUCTION

Semigroup is a set together with an associative binary operation. If a semigroup S has no identity element, then it can always be adjoined with one such, denoted by 1, and would be written : S^1 or simply, as $S = S^1$. By a $\mathcal{L}$-class of an element **a** in a semigroup S, we mean the set of all elements **b** of S which have the property : $S^1a = S^1b$. i.e. the principal left ideals generated by **a** and **b** coincide. Dually, we define a $\mathcal{R}$-class and put : $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$. Also, we define, a **J** b in S iff $S^1aS^1 = S^1bS^1$. These five relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$, **J** are due to Green and are equivalence relations over an arbitrary semigroup.

An element **a** of a semigroup S is said to be regular if there exists an element **b** of S such that : $aba = a$. If further, $bab = b$, then **b** is called a regular inverse or simply as an inverse of **a**. For instance, if $aba = a$, then a and bab are inverses of each other. A semigroup is termed as regular if each of its elements is regular. A regular semigroup in which the elements possess unique inverses is an 'inverse' semigroup. Inverse semigroups may also be characterised as regular semigroups whose idempotents (i.e. elements which equal their squares) form a semilattice (a commutative subsemigroup). 'Orthodox' semigroups are regular semigroups whose idempotents form subsemigroups. If **a** is a regular element of a semigroup S with **x** as an inverse, then ax is referred to as a left unit of **a** while xa is referred to as a right unit of **a** and the set of all inverses of **a** is denoted by $V(a)$. A regular semigroup in which every element has a unique left unit is called a 'right inverse' semigroup. It must be noted that in a regular semigroup the right and left units of each element coincide iff the semigroup is inverse and is a union of groups. For all other definitions, terminologies and notations used in what follows we refer to the treatises by Clifford and Preston (1961, 1967).

We extend some of, the above definitions, as follows :

Definitions — Suppose S is a semigroup and k denotes a positive integer. An element **a** of S is called k-regular if a^k is regular. S itself is called k-regular if the set $S^{(k)} = (s^k/s$ is in $S)$ is regular. S is called π-regular if every element of S is k-regular for some k depending on the element chosen. S is said to be k-orthodox (π-orthodox) is S is k-regular (π-regular) and the set E of idempotents of S is a subsemigroup. If x is a k-regular element of S, then any (regular) inverse of x^k in S is called a k-inverse of x. If every element of a k-regular (π-regular) semigroup has a unique k-inverse (π-inverse), then S is called a k-inverse (π-inverse) semigroup.

Likewise, other definitions may be extended.

Examples — (1) A regular semigroup is k-regular for every positive integer k.

(2) If S is the zero semigroup on a set with zero, then S is 2-regular but not regular, has a unique idempotent namely, zero but is not a group, etc.

(3) A left (right) zero semigroup is k-regular for every positive integer k but not k-inverse unless it is trivial. It is to be noted that : regular = 1 — regular.

(4) Example 5, p. 240 of Fuchs (1973) gives a π-regular ring which is not necessarily k-regular for any positive integer k.

(5) If $Z(p^k)$ is the cyclic group of order p^k, where p is a prime and k is a positive integer, $A = \oplus_p Z(p^k)$ where p ranges over all primes, then, S, the semigroup of endomorphism of A is k-regular and k-inverse, since it is commutative. (This is because any endomorphism of A is either an automorphism or is nilpotent of index at most k).

Remarks: (A) A left (right) cancellative semigroup is π-regular iff it is regular iff it is k-regular for each positive integer k iff it is a right (left) group.

(B) A cancellative semigroup is regular iff it is inverse iff it is π-regular, etc., and in each of such cases it reduces to a group. Consequently, a π-regular ring with identity without nonzero zero divisors is a skew-field.

(C) A one-sided cancellative inverse semigroup is a group. However, a one-sided cancellative k-inverse semigroup need not be a group for k exceeding 1 and would reduce to a group only when its idempotents commute.

A few easy results on regular semigroups are suitably modified to yield the following results 1.1 — 1.11 on π-regular semigroups, whose proofs are indicated. (S always denotes an arbitrary semigroup).

1.1. $a \in S$ is k-regular iff the $\mathcal{L}$-class ($\mathcal{R}$-class) of a^k contains an idempotent. (see Lemma 1.13, of Clifford and Preston 1961, 1967).

1.2. Suppose D is a $\mathcal{D}$-class in S, $a \in D$ and a is k-regular for some k. Then any $b \in D$ is k-regular, provided $b^k \in D$. In particular, if $D^{(k)}$ is a subset of D, then D is k-regular whenever it contains a regular element. (see Theorem 2.11 of Clifford and Preston 1961, 1967).

1.3. Let a be a k-regular element of S. Then

(i) Any k-inverse of **a** lies in D_{a^k}. i.e. $V(a^k)$ is a subset of D_{a^k}.

(ii) H_b contains a k-inverse of a iff both the $\mathcal{H}$-classes $R_{a^k} \cap L_b$ and $R_b \cap L_{a^k}$ contain idempotents.

(iii) No $\mathcal{H}$-class can contain more than one k-inverse of **a**.

(iv) **a** has a unique k-inverse in S iff the $\mathcal{L}$-class and the $\mathcal{R}$-class of a^k contain unique idempotents (cf. 1.5 below).

(for a proof, we refer to Theorem 2.18 of Clifford and Preston 1961, 1967).

1.4. If ρ is a congruence on a semigroup S and $x \lambda y$ in S for some elements x, y, then for every positive integer k, $x^k \rho y^k$.

1.5. $a^k \in S$ has a unique left unit iff R_{a^k} has a unique idempotent (see Theorem 3 of Bailes 1973).

1.6. If $(ef)^k = ef$ for all idempotents e, f in S, then S is k-orthodox if it is right (left) k-inverse.

1.7. Let S be any semigroup, U a π-regular subsemigroup of S and E the set of idempotents of S. Then

(i) for e, f in E, $e \mathcal{L} f$ in S iff $ef = e$, $fe = f$ and $e \mathcal{R} f$ in S iff $ef = f$, $fe = e$.

(ii) for a, b in U, $a^k \mathcal{L} b^m$ in U iff $a^k \mathcal{L} b^m$ in S and likewise, for $\mathcal{R}$.

(iii) for a, b in U, $a^k \mathcal{H} b^m$ in U iff $a^k \mathcal{H} b^m$ in S.

(see Result 9 of Hall 1969).

1.8. If S is π-orthodox (right π-inverse) then every π-regular subsemigroup of S is π-orthodox (right π-inverse).

1.9. An epimorphic image of a π-regular semigroup is π-regular. An epimorphic image of a k-inverse semigroup with $(ef)^k = ef$ for all idempotents e, f is again k-orthodox and k-inverse. In particular, if A is an ideal of S, then S is k-inverse iff both A and S/A are k-inverse, etc. (see Lemmas 7.35 and 7.36 of Clifford and Preston 1961, 1967).

1.10. Suppose S is a k-inverse semigroup in which $(ab)^k = a^k b^k$ for all a, b in S. (x^{-k} denotes the unique k-inverse of x in S). Let us define,

$$\rho = \{(x, y)/x^{-k}ex^k = y^{-k}ey^k, \text{ for all } e \text{ in } E, \text{ where } x, y \text{ are in } S\}.$$

Then ρ is the maximum idempotent-separating congruence on S and ρ^k is contained in $\mathcal{H}$. (see Lemma 7.57 of Clifford and Preston 1961, 1967).

1.11. Suppose $(ab)^k = a^k b^k$ for all a, b in S. If I is an ideal of S, then S is k-orthodox iff each of the semigroups I and S/I are k-orthodox. Consequently, S is k-orthodox iff each of its principal factors is k-orthodox. Moreover, any epimorphic image of S is k-orthodox.

2. Results on k-regular Semigroups

Theorem A — Suppose S is a semigroup and E is its set of idempotents. Let $(ef)^k = ef$ for all e, f in E where k is a positive integer. Then the following statements convey the same sense :

(1) S is k-regular and any two idempotents of S commute. (*Note* : In such a case S is necessarily k-orthodox).

(2) Every idempotent of S has a unique k-inverse, namely itself.

(3) S is a k-inverse semigroup.

(4) Every principal right ideal and every principal left ideal of S generated by elements of the form a^k, a in S have unique idempotent generators.

(5) R_{a^k} and L_{a^k} contain unique idempotents for every a in S.

(6) S is k-orthodox and E satisfies the identity : $efe = fef$.

PROOF : For a proof of the above theorem we refer to the proofs of Theorem 1.17 and Corollary 2.19 of Clifford and Preston (1961, 1967) and to the definitions.

Corollary — If S is π-orthodox, then any two idempotents of S commute iff S is π-inverse.

Theorem B — For a k-regular semigroup S in which $(ab)^k = a^k b^k$ for all a, b in S, the following are equivalent conditions :

(1) E^2 is a subset of E, where E denotes the set of idempotents of S.

(2) e is in E and x is a k-inverse of e implies that x is in E.

(3) a', b' are k-inverses of a, b in S imply that $b'a'$ is a k-inverse of ab.

PROOF : This is a generalized version of Lemma 1.3 of Reilly and Schlieblich (1967) for k greater than 1.

Similarly Theorem 1.5 and Corollary 1.6 of Reilly and Schlieblich (1967) on generalization yield :

Theorem C — Suppose S is a semigroup in which $(ab)^k = a^k b^k$ for all a, b in S and B is an idempotent subsemigroup of S. Then the largest k-orthodox subsemigroup of S with B as its band of idempotents is given by

$$B^* = \{x \in S/\text{for some } y^k, y^k \text{ is a } k\text{-inverse of } x,\ x^k y^k,\ y^k x^k \in B \text{ and } x^k B y^k \subset B,\ y^k B x^k \subset B\}.$$

Corollary — Suppose S is a semigroup in which $(ab)^k = a^k b^k$ holds for all a, b in S and B a band contained in the centre of S. Then, B^* as constructed in Theorem C is the largest k-inverse subsemigroup of S with B as its semilattice of idempotents.

3. IDEMPOTENT-SEPARATING CONGRUENCES

Suppose S is π-orthodox and E is its band of idempotents. Clearly, by McLean's result, E is a semilattice of rectangular bands and hence, $E = U_{p \varepsilon Y} E_p$, where each E_p is a rectangular band, Y is a semilattice, for all p, q in Y : $E_p \cap E_q$ is null if $p \neq q$ and $E_p E_q \subset E_{pq}$. If e is in E_p, we write E_p as $E(e)$. Clearly, for e, f in E, we have : $E(e)\, E(f)$ to be a subset of $E(ef) = E(fe)$ and $E(e) = V(e)$, the set of inverses of e in S. We refer to Hall (1970) for further details and make use of the following.

Lemma 3.1 — If ρ is a congruence on a π-orthodox semigroup S, then a ρ-class which is a subsemigroup contains an idempotent. Hence an epimorphic image of a π-orthodox semigroup is π-orthodox.

Lemma 3.2 — Let S be π-orthodox and define $\rho = ((ef, fe) \in S \times S/e, f \in E)$. Then, ρ^*, the congruence generated by ρ is the finest π-orthodox congruence on S such that the idempotents of S/ρ^* commute. Further, if η is any congruence on S containing ρ^*, then S/η is π-orthodox and its idempotents commute. Hence, S/ρ^* and S/η are π-inverse semigroups.

Lemma 3.3 — Let S be k-regular and $(ef)^k = ef$ for all e, f in E. Then S is k-orthodox iff for all a, b in S, $V(a^k) \cap V(b^k)$ is nonempty implies that : $V(a^k) = V(b^k)$. In fact, S is k-orthodox iff for all e, f in E, $V(e)$ has a nonnull intersection with $V(f)$ implies that : $V(e) = V(f)$.

These lemmas lead to the main result analogous to Theorem 3 of Hall (1970) :

Theorem D — Suppose S is k-regular and $(ab)^k = a^k b^k$ for all a, b in S. Then,

$$\mathcal{Y} = \{(x, y) \in S \times S/V(x^k) = V(y^k)\}$$

is the finest k-inverse congruence on S iff S is k-orthodox.

Suppose S is k-orthodox and $(ab)^k = a^k b^k$ for all a, b in S. Let T_X denote the set of all transformations on a set X and T_X^* the dual of T_X. The set of $\mathcal{R}$-classes and $\mathcal{L}$-classes of E are : $E/\mathcal{R} = (eE(e)/e$ is in $E)$ and $E/\mathcal{L} = (E(e)\, e/e$ is in $E)$. For each **a** in S, define : $\lambda_{a^k} \in T_{E/\mathcal{R}}$, $\rho_{a^k} \in T_{E/\mathcal{L}}$ by :

$$eV(e)\,\lambda_{a^k} = a^k e V(a^k e) \text{ [i.e. } R_e \lambda_{a^k} = R_{a^k e a'},\ a' \in V(a^k)]$$

$$V(e)\, e\rho_{a^k} = V(ea^k)\, ea^k \text{ [i.e. } L_e \rho_{a^k} = L_{a' e a^k},\ a' \in V(a^k)]$$

for all e in E. Clearly, $\lambda_{b^k} \cdot \lambda_{a^k} = \lambda_{(ab)^k}$ and $\rho_{a^k} \cdot \rho_{b^k} = \rho_{(ab)^k}$. Also, if η is a congruence on S, we write : $\eta^k = \{(a^k, b^k)/(a, b)$ is in $\eta\}$ and we denote $(x^k/x$ is in $S)$ by $S^{(k)}$. It is clear that $S^{(k)}$ is a semigroup whenever S is a semigroup in which $(ab)^k = a^k b^k$ is valid for all a, b in S.

Generalizing Theorems 1 and 2 of Hall (1969) and Theorem 4.4 of Meakin (1971) we obtain the following results.

Theorem E — Suppose S is k-orthodox and the elements of S satisfy : $(ab)^k = a^k b^k$, for arbitrary a, b in S. For **a** in S, define : $\xi(a) = (\lambda_{a^k}, \rho_{a^k})$. Then $S \to T_{E/\mathcal{R}}^* \times T_{E/\mathcal{L}}$ is a homomorphism and $\mu = \xi \circ \xi^{-1}$ is the maximum idempotent-separating congruence on S. The latter may also be characterized by :

$$\mu = \left\{(a, b) \in S \times S \Big/ \begin{array}{r} \text{there exists } a' \text{ in } V(a^k), b' \text{ in } V(b^k) \text{ such that for all} \\ e \text{ in } E, a^k e a' = b^k e b' \text{ and } a' e a^k = b' e b^k \end{array}\right\}.$$

Further, $\mu^k \subset \mathcal{H}$.

Theorem F — Suppose S is k-orthodox in which the elements satisfy: $(ab)^k = a^k b^k$. Then, $S^{(k)} \to T^*_{E/\mathcal{R}} \times S/\mathcal{Y} \times T_{E/\mathcal{L}}$ defined by: $a^k \to (\lambda_{a^k}, a^k\mathcal{Y}, \rho_{a^k})$ is a monomorphism.

Suppose $\wedge(S)$ denotes the complete lattice of all congruences on a semigroup S and

$$\Sigma^k(\mathcal{H}) = \{\rho \in \wedge(S) / \rho^k \subset \mathcal{H}\},\ k \geqslant 1.$$

Then following Munn (1964) we obtain the results given below.

Lemma 3.4 — Let a be a k-regular element of S, with a' in $V(a^k)$, $e = a^k a'$ and $\rho \in \Sigma^k(\mathcal{H})$. Then, $e\rho^k$ is a normal subgroup of the group H_e and for any b in H_{a^k}, (a, b) is in ρ iff $b^k a'$ belongs to $e\rho^k$. Furthermore, for any, $\rho, \sigma \in \Sigma^k(\mathcal{H})$, we have: $\rho \circ \sigma = \sigma \circ \rho$ whenever S is k-regular.

Theorem G — Suppose S is k-regular. Then $\Sigma^k(\mathcal{H})$ is precisely the set of all idempotent-separating congruences on S. Further, $\Sigma^k(\mathcal{H})$ is a modular sublattice of commuting congruences of $\wedge(S)$ with a greatest and a least element.

Also a generalization of Theorem 3.4 of Reilly and Schlieblich (1967) yields the following result.

Theorem H — Suppose S is k-regular and

$$\theta = \{(\rho, \sigma) \in \wedge(S) \times \wedge(S) / \rho \mid_E = \sigma \mid_E\}.$$

Then θ is a meet compatible equivalence on $\wedge(S)$ and each θ-class is a complete modular sublattice of $\wedge(S)$ with a greatest and a least element.

4. Right k-Inverse Semigroups

A k-regular semigroup S is said to be right k-inverse whenever for every a in S, a', a'' are in $V(a^k)$ imply that: $a^k a' = a^k a''$. i.e. every element has a unique left k-unit. Many equivalent characterizations of such semigroups is given below. (cf. Theorem I).

In this section, we make use of the results from, Bailes (1973), Venkatesan (1972), Warne (1972), Yamada (1973) and Harinath (1977) with slight changes in the notations. Theorems 15, 17, 19 of Bailes (1973), Result 9 of Hall (1969) and Theorem D obtained above together yield the following results.

Proposition 4.1 — A homomorphic image of a right k-inverse semigroup S whose elements satisfy: $(ab)^k = a^k b^k$, is again right k-inverse satisfying a similar identity.

Proposition 4.2 — Suppose S is k-regular and we define,

$$\mathcal{S}^* = \{(e, f) \in E \times E / ef = f,\ fe = e\} = \mathcal{R} \mid_E$$

where E is the set of idempotents of S. If $\mathcal{S}$ is the smallest congruence on S containing $\mathcal{S}^*$, then $\mathcal{S}$ is the finest congruence on S such that $S/\mathcal{S}$ is a right k-inverse semigroup.

Proposition 4.3 — Suppose S is right k-inverse and $(ab)^k = a^k b^k$ for all a, b in S. Then $\mathcal{Y}$ is the finest congruence on S containing the Green's relation $\mathcal{L}$. (In this case, S is k-orthodox).

Proposition 4.4 — A congruence-free right k-inverse semigroup is either a congruence-free k-inverse semigroup or a 2-element left zero semigroup.

Corollary — A congruence-free k-orthodox semigroup is either a congruence-free k-inverse semigroup or a congruence-free rectangular band.

Remark : Suppose S is a π-regular semigroup with E as its set of idempotents. Then $ef = fe$ holds in E iff $efe = fef$, if either S is π-orthodox or if S has an identity. In case S has an identity, then the following ten identities on E are equivalent :

(1) $ef = fe$, (2) $efe = fef$, (3) $efg = egf$, (4) $efg = feg$,

(5) $efgh = egfh$, (6) $efge = egfe$, (7) $efgh = efhg$,

(8) $efgh = fegh$, (9) $efege = egefe$, (10) $efege = efge$., etc.

In particular, the set of idempotents of a π-regular semigroup with identity satisfies a permutation identity iff it satisfies commutativity, in which case it becomes a π-inverse monoid.

Many equivalent characterizations of right inverse semigroups are generalized to yield :

Theorem I — Suppose S is k-regular in which the elements satisfy : $(ab)^k = a^k b^k$. Then the following statements characterize right k-inverse semigroups (cf. Harinath 1977) :

(1) S is right k-inverse (in such a case, S would be k-orthodox).

(2) Every R_{a^k} for **a** in S contains a unique idempotent.

(3) Every element of S has a unique left k-unit.

(4) E, the band of idempotents of S, is a semilattice of left zero semigroups.

(5) E is a left regular band. i.e. $efe = ef$, for all e, f in E.

(6) $L_e = V(R_e^k) = \{x \in S / x \in V(y^k)$ for some $y \in R_e\}$ for every e in E.

(7) $Se \cap Sf = Sef = Sfe$ for any two idempotents e, f in S.

(8) S is the right-half direct product of a right regular band with a k-inverse semigroup.

(9) Given e, f in E, there exists a g in E such that : $ef = ge$.

(10) Given any **a** in S, we can find a **b** in S such that : $V(a^k) \subset L_b$.

5. New Results on Right Inverse Semigroups

In this concluding section, we note a few results on right inverse semigroups which are not already noted in the literature. These may be extended to include right k-inverse semigroups and for the sake of brevity, the details are omitted.

Proposition 5.1 — Any right inverse semigroup may be embedded in a simple right inverse semigroup with identity.

PROOF : Theorems 8.45 and 8.48 of Clifford and Preston (1961, 1967).

Remark : There exist simple right inverse semigroups with identity having an arbitrary number of $\mathcal{D}$-classes. (cf. Corollary 8.50 of Clifford and Preston 1961, 1967).

Proposition 5.2 — A left cancellative right inverse semigroup is a group.

PROOF : Follows from the equivalence (1) $\Leftrightarrow$ (5) of Theorem I for $k = 1$.

Note : A right cancellative regular semigroup is a left group and hence right inverse. In the presence of an identity, this reduces to a group.

Proposition 5.3 — Suppose S is regular and right cancellative. i.e, S is a left group. If E is the band of idempotents of S, then the maximum-idempotent-separating congruence ρ on S is given by :

$$\rho = \{a, b) \in S \times S / aea' = beb' \text{ for some } a' \text{ in } V(a) \text{ and } b' \text{ in } V(b)\}.$$

PROOF : We refer to Meakin (1971) and Reilly and Schlieblich (1967) for details.

Proposition 5.4 — Suppose S is orthodox and $\mathcal{S}$ is the finest congruence on S containing $\mathcal{R}(\mathcal{L})$. Then $\mathcal{S}$ is the smallest congruence on S such that $S/\mathcal{S}$ is a right (left) inverse semigroup.

PROOF : It follows from Theorem 17 of Bailes (1973) and Result 9 of Hall (1969).

Corollaries — (A) If S is right inverse and a union of groups, then $\mathcal{L} = \mathcal{D} = \mathbf{J}$ is a congruence on S and. $S/\mathcal{L}$ is an inverse semigroup, being a semilattice of groups.

(B) If S is right inverse and a union of groups, then $\mathcal{L}$ is a congruence on S and S is a band of maximal left groups.

More generally, we have (the rephrased version of the above corollaries as) :

Proposition 5.5 — If S is right inverse, then S is a union of groups iff $V(a)$ is a subset of L_a for each a in S, which in turn is true iff $\mathcal{D} = \mathcal{L}$. In such a case, $\mathcal{L}$ is a congruence, S is a band of maximal left groups and $S/\mathcal{L}$ is an inverse semigroup.

Suppose S is a right inverse semigroup and E is its left normal band. Define a subset @ of $E \times E$ by : $@ = \{(e, f)/eE$ and fE are isomorphic$\}$. Then @ is nonempty since (e, e) is in @ for every e in E and it is possible that @ can be equal to $E \times E$. Following the main theorem of Hall (1969) or directly, we can arrive at the validity of the following results.

Theorem J — A right inverse semigroup S with E as its band of idempotents is bisimple iff eE and fE are isomorphic for all e, f in E or equivalently, if and only if, $@ = E \times E$.

Theorem K — A right inverse semigroup S with E as its band of idempotents is a union of groups iff $@ = U_{e \epsilon E} V(e) \times V(e) \subset E \times E$, where, $E = U_{e \epsilon E} V(e)$. In such a case, S is a semilattice of left simple semigroups, $\mathcal{L} = \mathcal{D} = \mathbf{J}$, $\mathcal{R} = \mathcal{H}$, etc.

Note : A right (left) inverse semigroup is a band of groups iff $\mathcal{R}(\mathcal{L}) = \mathcal{H}$ is a congruence.

ACKNOWLEDGEMENT

The author is indebted to Prof. K. N. Kamalamma for guidance and encouragement in preparing this paper.

REFERENCES

Bailes, G. L. (Jr.) (1973). Right inverse semigroups. *J. Alg.*, **26**, 492-507.

Clifford, A. H., and Preston, G. B. (1961, 1967). Algebraic Theory of Semigroups, Volumes I and II. *AMS Math. Surveys, No.* 7.

Fuchs, L. (1973). Infinite Abelian Groups, Volume II. *Academic Press*, New York.

Hall, T. E. (1969). On regular semigroups whose idempotents form a subsemigroup. *Bull Austr. math. Soc.*, **1**, 195-208.

———— (1970). On orthodox semigroups and uniform and antiuniform bands. *J. Alg.*, **16**, 204-17.

Harinath, K. S. (1977). On a generalization of inverse semigroups. *Indian. J. pure appl. Math.*, **8**, 166-178.

Meakin, J. C. (1971). Congruences on orthodox semigroups. *J. Austr. math. Soc.*, **12**, 323-41.

Munn, W. D. (1964). A certain sublattice of the lattice of congruences on a regular semigroup. *Proc. Camb. phil. Soc.*, **60**, 385-91.

Reilly, N. R., and Schlieblich, H. E. (1967). Congruences on regular semigroups. *Pacific J. Math.*, **23**, 349-60.

Rhodes, J. (1971). Applications of automata theory and algebra. *Pre-print.*

Venkatesan, P. S. (1972). Bisimple left inverse semigroups. *Semi. Forum*, **4**, 34-45.

Warne, R. J. (1972). *L*-unipotent semigroups. *Nige. J. Sci.*, **5**, 245-48.

Yamada, M. (1973). Orthodox semigroups whose idempotents satisfy a certain identity. *Semi. Forum*, **6**, 113-28.