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ON SPHERICALLY SYMMETRIC SOLUTIONS OF A SCALAR-TENSOR THEORY OF GRAVITATION IN A LYRA MANIFOLD

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A result having formal similarity to Birkhoff's theorem in general relativity has been proved in the presence of electromagnetic fields for a scalar-tensor theory of gravitation when the scalar field is independent of time.

1. INTRODUCTION

Recently Sen and Dunn (1971) have proposed a new scalar-tensor theory of gravitation, as an alternative to Brans-Dicke theory (1961), in which both the scalar and tensor fields have intrinsic geometrical significance. The scalar field in this theory is characterised by the function $\Phi = \Phi(x^i)$ where x^i are coordinates in the four-dimensional Lyra manifold, and the tensor field is identified with the metric tensor of the manifold. But Jeavons *et al.* (1975) have pointed out that the field equations proposed by Sen and Dunn (1971) are although simple they are not derivable from the usual variational principle. The field equations proposed by Jeavons *et al.* (1975) are

$$R_{ij} - \tfrac{1}{2} g_{ij} R + \Phi^{-1}(\Phi_{i;j} - g_{ij} \Box \Phi) - \omega \Phi^{-2}(\Phi_{,i}\Phi_{,j} - \tfrac{1}{2} g_{ij}\Phi_{,k}\Phi^{,k}) = -\Phi\, T_{ij} \qquad \ldots(1)$$

where $\omega = \frac{3}{2}$, T_{ij} is the material energy-momentum tensor and R_{ij} and R are respectively the usual Ricci tensor and Riemann curvature scalar (in our units $c = 8\pi G = 1$). Singh and Rai (1977) have shown that in the vacuum case of the present theory, the time-independence of the scalar field implies that of the gravitational and electromagnetic fields. In this paper, following Das (1960), we have shown that this result is valid for this theory in presence of the electromagnetic fields also.

2. SPHERICALLY SYMMETRIC FIELDS IN THE SCALAR-TENSOR THEORY

Birkhoff's theorem states that every spherically symmetric solution of the Einstein vaccum field equations is static (Birkhoff 1927). Das (1960) has extended this theorem to the combined electromagnetic and gravitational fields in general relativity. Reddy (1977) has proved a similar theorem for the Sen-Dunn scalar-tensor theory under the very strong condition that the scalar field is independent of time. In this paper we show that a result of the type proved by Reddy holds for spherically symmetric fields in the scalar-tensor theory of Jeavons *et al.* (1975) also when the scalar field is time-independent.

In the notation of Das (1960), the field equations due to Jeavons *et al.* (1975) in the presence of electromagnetic field are

$$D^i \equiv F^{ij}_{;j} = 0 \qquad \text{...(2a)}$$

$$E^i \equiv \tfrac{1}{2}\,\epsilon^{jkli} F_{kl,j} = F_{[kl,i]} = 0 \qquad \text{...(2b)}$$

$$Q^i_j \equiv R^i_j - \tfrac{1}{2}\delta^i_j R - \frac{\omega}{\Phi^2}\left(\Phi^{,i}\Phi_{,j} - \tfrac{1}{2}\delta^i_j \Phi_{,k}\Phi^{,k}\right)$$

$$+ \frac{1}{\Phi}\left(\Phi^{;i}_{;j} - \delta^i_j \Box \Phi\right) + \Phi E^i_j = 0 \qquad \text{...(2c)}$$

where comma and semicolon denote partial and covariant derivatives respectively and E^i_j is defined as

$$E^i_j = F^{ik}F_{jk} - \tfrac{1}{4}\delta^i_j F^{kl}F_{kl}$$

We consider the spherically symmetric metric in the form

$$ds^2 = e^{\beta}\, dt^2 - e^{\alpha}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad \text{...(3)}$$

where $\alpha = \alpha(r, t)$ and $\beta = \beta(r, t)$. On account of spherical symmetry

$$F_{12} = F_{13} = F_{24} = F_{34} = 0;\; F_{14};\; F_{23} \neq 0 \qquad \text{...(4)}$$

Also spherically symmetry implies that $\Phi = \Phi(r, t)$.

In view of (4), the field eqn. (2*a*) and (2*b*) yield (Das 1960),

$$F_{14} = \left(\frac{\epsilon}{r^2}\right) e^{(\alpha+\beta)/2}, \quad F_{23} = \mu \sin\theta \qquad \text{...(5)}$$

where ϵ and μ are constants of integration which can physically be interpreted as the electric charge and magnetic pole strength respectively, of a point source.

Now using (4), (5) and the metric (3) the field eqns. (2*c*) can be written as

$$Q^1_1 + Q^4_4 = \left\{1 + \frac{r}{2}(\beta' - \alpha') - \frac{r^2}{2\Phi}\left(\Phi'' - \frac{\Phi'\alpha'}{2} + \frac{4\Phi'}{r} + \frac{\Phi'\beta'}{2}\right)\right.$$

$$\left. - e^{\alpha}\left[1 - \frac{\Phi}{r^2}\left(\frac{\epsilon^2 + \mu^2}{2}\right) - \frac{r^2}{2\Phi} e^{-\beta}(2\ddot{\Phi} - \dot{\Phi}\dot{\beta} + \dot{\Phi}\dot{\alpha})/2\right]\right\} = 0 \qquad \text{...(6)}$$

$$Q_2^2 = Q_3^3 = -e^{-\alpha}\left(\frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} + \frac{(\beta'-\alpha')}{2r}\right)$$

$$+ e^{-\beta}(2\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\beta})/4$$

$$+ \frac{\Phi}{2r^4}(\epsilon^2 + \mu^2) + \frac{\omega}{2\Phi^2}(e^{-\beta}\dot{\Phi}^2 - e^{-\alpha}\Phi'^2)$$

$$+ \frac{1}{\Phi}\left\{e^{-\alpha}\left(\Phi'' - \frac{\Phi'\alpha'}{2} + \frac{\Phi'}{r} + \frac{\Phi'\beta'}{2}\right)\right.$$

$$\left. - e^{-\beta}(2\ddot{\Phi} - \dot{\Phi}\dot{\beta} + \dot{\Phi}\dot{\alpha})/2\right\} = 0 \qquad \text{...(7)}$$

$$Q_4^1 = -\frac{\dot{\alpha}}{r}e^{-\alpha} + \frac{\omega}{\Phi^2}(\dot{\Phi}\Phi' e^{-\alpha}) + (e^{-\alpha}/\Phi).(\Phi'\dot{\alpha} + \dot{\Phi}\beta' - 2\dot{\Phi}')/2 = 0 \qquad \text{...(8)}$$

$$Q_1^4 = \frac{\dot{\alpha}}{r}e^{-\beta} - \frac{\omega}{\Phi^2}(\dot{\Phi}\Phi' e^{-\beta}) + (e^{-\beta}/\Phi)(2\dot{\Phi}' - \dot{\Phi}\beta' - \Phi'\dot{\alpha})/2 = 0 \qquad \text{...(9)}$$

$$Q_1^1 = -e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{\omega}{2\Phi^2}$$

$$\times (\Phi'^2 e^{-\alpha} + \dot{\Phi}^2 e^{-\beta}) - \frac{\Phi}{2r^4}(\epsilon^2 + \mu^2)$$

$$+ \frac{1}{\Phi}\left\{e^{-\alpha}\left(\frac{2\Phi'}{r} + \frac{\Phi'\beta'}{2}\right) - e^{-\beta}\left(\ddot{\Phi} - \frac{\dot{\Phi}\dot{\beta}}{2}\right)\right\} = 0 \qquad \text{...(10)}$$

where primes and dots denote partial derivatives with respect to r and t respectively.

It can be seen that when Φ is a constant the above set of field equations reduces to the Einstein-Maxwell field equations for which the Birkhoff theorem follows as shown by Das (1960). When the electromagnetic field is absent in this scalar-tensor theory, the time-independence of Φ implies the time-indendence of g_{ij} also.

When the scalar field Φ is independent of t, that is

$$\dot{\Phi} = 0 \qquad \text{...(11)}$$

from (8) and (9) we have

$$\dot{\alpha}\left(\frac{1}{r} - \frac{\Phi'}{2\Phi}\right) = 0$$

which implies that either

$$\dot{\alpha} = 0 \qquad \text{...(12)}$$

or $$\frac{\Phi'}{\Phi} = \frac{2}{r} \text{ i.e. } \Phi = \Phi_0 r^2, \; \Phi_0 = \text{constant.} \qquad ...(13)$$

When $\Phi = \Phi_0 r^2$, from (6) and (10), we have

$$e^{\alpha} = 0. \qquad ...(14)$$

Therefore in this case no solution exists. When $\dot{\alpha} = 0$, eqn. (6) reduces to

$$1 + \frac{r}{2}(\beta' - \alpha') - \frac{r^2}{2\Phi}\left(\Phi'' - \frac{\Phi'\alpha'}{2} + \frac{4\Phi'}{r} + \frac{\Phi'\beta'}{2}\right)$$

$$= e^{\alpha}\left\{1 - \frac{\Phi}{r^2}\left(\frac{\epsilon^2 + \mu^2}{2}\right)\right\}. \qquad ...(15)$$

Differentiation of (15) with respect to t along with the use of (11) and (12) gives

$$\dot{\beta}'\left(\frac{\Phi'}{\Phi} - \frac{2}{r}\right) = 0.$$

Since for $\Phi'/\Phi = 2/r$ no solution exists, we have

$$\dot{\beta}' = 0. \qquad ...(16)$$

From this one has

$$\beta = F(r) + G(t)$$

where F and G are arbitrary function of r and t respectively. Introducing a time coordinate transformation $dt' = e^{G(t)/2}dt$ it follows, in view of (12), that the metric (3) is static. Hence time-independence of the scalar field is a sufficient condition to make the spherically symmetric g_{ij} and F_{ij} time independent in the scalar-tensor theory of Jeavons *et al.* (1975)

3. Conclusions

It has been shown that in the presence of electromagnetic fields in the scalar-tensor theory proposed by Jeavons *et al.* (1975) when the scalar field is independent of time the spherically symmetric gravitational and electromagnetic fields become static. This leads to the conclusion that, possibly, the interaction of the time-dependent scalar field with the electromagnetic field may produce electromagnetic monopole radiation.

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