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ON THE SPACE OF A CERTAIN CLASS OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Based on the definition of order and type of entire functions of several complex variables due to Dzrbasian (1955), spaces of entire functions of finite order and finite type are introduced. These spaces are noted to be complete linear metric spaces. Besides finding out the duals of these spaces, the proper bases are characterised in terms of growth conditions. A theorem dealing with the convergence property of sequences in these spaces is given at the end.

§1. In a recent paper, Kamthan (1973) has made a study on the space of entire functions of several complex variables. The object of the present paper is to investigate some properties of the spaces of entire functions of finite order and type based on the definitions of order and type of entire functions of several complex variables due to Dzrbasian (1955). For the sake of simplicity we shall consider the case of two complex variables, since our results can be easily extended to any finite number of variables.

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z_1^m z_2^n$$

is an entire function if and only if

$$| a_{mn} |^{1/(m+n)} \to 0 \text{ as } m + n \to \infty.$$

§2. Let $X(\rho, d)$ stand for the class of entire functions $f : \mathcal{S}^2 \to \mathcal{S}$ where

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z_1^m z_2^n, \; a_{mn} \in \mathcal{S}$$

and $\mathcal{S}$ is the complex plane such that

$$\limsup_{r \to \infty} \frac{\log M_f(r)}{r^\rho} \leqslant d, \qquad \ldots(2.1)$$

where $M_f(r) = | f(z_1, z_2) |$

subject to $r = | z_1 | + | z_2 |$ and $\rho = \limsup \log \log M_f(r)/\log r$. According to Dzrbasian (1955), $f(z_1, z_2)$ is of order ρ and type d at the most if and only if the following condition is satisfied

$$\limsup_{m+n\to\infty} (m+n)^{(1/\rho)-1} [m^m n^n \mid a_{mn} \mid]^{1/(m+n)} \leqslant (\rho de)^{1/\rho}. \qquad \ldots(2.2)$$

Let us write

$$\lambda(m, n; \rho, d) = \frac{(m+n)^{((1/\rho)-1)(m+n)}\, m^m n^n}{(\rho de)^{(m+n)/\rho}}. \qquad \ldots(2.3)$$

By using (2.2), for every $\delta > 0$, the double series

$$\| f, d+\delta \| = \mid a_{00} \mid + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mid a_{mn} \mid \lambda(m, n; d+\delta),$$

$$m+n \neq 0,\ m, n \geqslant 0$$

converges for $f \in X(\rho, d)$ and $\| f, d+\delta \|$ defines a norm on $X(\rho, d)$. Let us denote the corresponding normed space by $X(\rho, d, \delta)$. The lattice product of these normed topologies denoted by $X(\rho, d)$. The space $X(\rho, d)$ is metrisable and its metric is given by

$$\| f \| = \sum_{p=1}^{\infty} \frac{\| f, d+(1/p) \|}{2^p [1 + \| f, d+(1/p) \|]}.$$

With this topology on $X(\rho, d)$, we can prove the following two theorems as proved in Kamthan and Gupta (1974). Let us write $\delta_{mn} = z_1^m z_2^n$.

Theorem 1 — Let $f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}\delta_{mn} \in X(\rho, d)$. Then the sequence of partial sums of the series for f converges to f in $X(\rho, d)$.

Theorem 2 — The metric space $X(\rho, d)$ is a complete metric space.

§3. In this section we shall find the form of a continuous linear functional on $X(\rho, d)$ and $X(\rho, d, \delta)$ for each $\delta > 0$. It is well known that $X^*(\rho, d)$ is the union of $X^*(\rho, d, \delta)$ for $\delta > 0$. Hence it is enough, if we find the set of all continuous linear functionals on $X(\rho, d, \delta)$ which is given in the form of the following theorem for which the proof is similar to the corresponding theorem of Kamthan and Gupta (1974).

Theorem 3 — (i) A continuous linear functional $\varphi \in X^*(\rho, d, \delta)$ is of the form $\varphi(f) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}c_{mn}$, where $f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z_1^m z_2^n \in X(\rho, d)$ and

$$c_{mn}/\lambda(m, n; \rho, d+\delta)$$

is bounded and conversely.

(ii) Every $\varphi \in X^*(\rho, d)$ is of the form $\varphi(f) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} a_{mn}$ where $f \in X(\rho, d)$ and $\limsup_{m+n\to\infty} (m+n)^{(1/\rho)-1} [m^m n^n \mid a_{mn} \mid]^{1/(m+n)} < (\rho e d)^{1/\rho}$.

§4. In this section we shall study proper bases in $X(\rho, d)$. Let $\{\alpha_{mn}, m, n \geqslant 0\}$ be a double squence of entire functions in $X(\rho, d)$. The sequence $\{\alpha_{mn}\}$ is said to be linearly independent if $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn} = 0$ implies $c_{mn} = 0$ for all sequences $\{c_{mn}\}$ of complex numbers for which the series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ converges in $X(\rho, d)$. $\{\alpha_{mn}\}$ spans a subspace X_0 of $X(\rho, d)$ provided X_0 consists of all linear combinations $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ which are convergent in $X(\rho, d)$. We shall adopt the following definition of proper bases introduced by Arsove (1958).

Definition — A basis $\{\alpha_{mn}\}$ in a subspace of $X(\rho, d)$ is called a proper basis, provided $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ converges if and only if $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\delta_{mn}$ converges for all sequences $\{c_{mn}\}$ of complex numbers. The convergence in $X(\rho, d)$ is equivalent to the condition

$$\limsup_{m+n\to\infty} (m+n)^{(1/\rho)-1} [m^m n^n \mid a_{mn} \mid]^{1/(m+n)} < (\rho e d)^{1/\rho}.$$

To characterise the proper bases, we need the following two lemmas. Since the proofs of the lemmas run parallel to those given by Kamthan and Gupta (1974), the proofs are substantially shortened.

Lemma 1 — The following properties are equivalent

(I) $$\limsup_{m+n\to\infty} \frac{\| \alpha_{mn}; d+\delta \|^{1/(m+n)}}{(m^m n^n)^{1/(m+n)} (m+n)^{(1/\rho)-1}} \leqslant \frac{1}{(d\rho e)^{1/\rho}}, \text{ for each } \delta > 0.$$

(II) For all sequences $\{c_{mn}\}$ of complex numbers $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\delta_{mn}$ converges in $X(\rho, d)$ implies $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ converges in $X(\rho, d)$.

(III) For all sequences $\{c_{mn}\}$ of complex numbers $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ converges in $X(\rho, d)$ implies $c_{mn}\alpha_{mn} \to 0$ in $X(\rho, d)$.

PROOF : (I) ⇒ (II) and (II) ⇒ (I) are obvious. So the proof of the theorem is complete, if we show that (III) ⇒ (I). To this end, assume that (III) is true and I is not true. Assuming condition (I) is not satisfied, we construct a sequence $\{c_{mn}\}$ as follows

$$c_{mn} = \begin{cases} \dfrac{1}{\| \alpha_{mn};\, d + \delta \|}, & \text{when } m = m_k,\ n = n_k. \\ 0 \text{ otherwise.} \end{cases}$$

One can verify as in Kamthan and Gupta (1974) that c_{mn} defined above satisfies (2.2), but $\| c_{mn}\alpha_{mn};\, d + \delta \| = 1$ which does not tend to zero in $X(\rho, d, \delta)$ contradicting the hypothesis that $c_{mn}\alpha_{mn} \to 0$ as $m + n \to \infty$ in $X(\rho, d)$. This proves (III) ⇒ (I).

Lemma 2 — The following three properties are equivalent

(I′) $$\lim_{\delta \to 0} \left\{ \liminf_{m+n\to\infty} \frac{\| \alpha_{mn};\, d + \delta \|^{1/(m+n)}}{[m^m n^n]^{1/(m+n)}\,(m+n)^{(1/\rho)-1}} \right\} \geqslant \frac{1}{(\rho e \delta)^{1/\rho}}$$

(II′) For all sequences $\{c_{mn}\}$ of complex numbers $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ converges in $X(\rho, d)$ implies $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\delta_{mn}$ converges in $X(\rho, d)$.

(III′) For all sequences of complex numbers c_{mn}, $c_{mn}\alpha_{mn} \to 0$ in $X(\rho, d)$ implies $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\delta_{mn}$ converges in $X(\rho, d)$.

PROOF : (III′) ⇒ (II′) and (I′) ⇒ (III′) can be proved without difficulty. To prove (II′) ⇒ (I′), first assume (II′) is true and (I′) is not true. Using the fact that (I′) is not true, we can construct a sequence $\{c_{mn}\}$ for a subsequence of values of m and n,

$$c_{mn} = \begin{cases} \dfrac{[(d + \eta)\, \rho e]^{(m+n)/\rho}}{m^m n^n (m + n)^{(m+n)(1-\rho)/\rho}}, & \text{when } n = n_k,\ m = m_k \\ 0 \text{ otherwise.} \end{cases}$$

We can easily verify through Kamthan and Gupta (1974) that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$ converges in $X(\rho, d)$ but $\{c_{mn}\}$ does not satisfy (2.2). This contradiction proves that (II′) ⇒ (I′).

In the above (II) and (II′) imply for a sequence $\{\alpha_{mn}\}$ to be a proper base. Hence combining the above two lemmas, the following is the characterization of proper basis in $X(\rho, d)$.

Theorem 4 — A basis $\{\alpha_{mn}\}$ in a closed subspace X_0 of $X(\rho, d)$ is proper if and only if the conditions (I) and (I′) hold.

§5. Convergence in $X(\rho, d)$ can be explained with the help of relative uniform covergence as follows.

The topology on X is the topology of uniform convergence on compact subsets $\mathcal{S}^2$ (Kamthan 1973). But such a result is not true in this case as shown by the following example.

Example. Consider $f_{mn} = (1/(m+n))\, e^{z_1+z_2}$ so that we have

$$a_{mn} = \left(\frac{1}{m!} \cdot \frac{1}{n!}\right) \frac{1}{m+n}.$$

Since $| a_{mn} |^{1/(m+n)} \to 0$ as $m+n \to \infty$, $\{f_{mn}\}$ is a sequence of entire functions in z_1 and z_2. According to the definition of order and type introduced by Dzrbasian (1955), $\rho = 1$ and $d = 1$ so that $f_{mn} \in X(\rho, d)$, where $X(\rho, d) = X(1, 1)$. Simple calculations lead to

$$\lambda(m, n : 1, 1) = \frac{m^m n^n}{e^{m+n}} \quad \text{and}$$

$$\| f_{mn} : 1 \| = \frac{1}{m+n} + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m!\, n!} \frac{m^m n^n}{e^{m+n}}.$$

Since the series on the right side of the above equation is not convergent, $\{f_{mn}\}$ is not convergent in $X(1, 1)$. But on compact subsets of the complex plane $\mathcal{S}^2$, $f_{mn}(z_1, z_2)$ tends to zero as $m+n$ tends to infinity, as it can be verified easily.

However we can use the notion of relative uniform convergence. To state the result precisely, we need the following preliminaries.

For each positive number $\delta > 0$ and for each $f \in X(\rho, d)$, let us define

$$\| f \| = \text{Max} \{e^{-(d+\delta)r^\rho} \, | f(z_1, z_2) | \}, \quad \text{where} \quad r = r_1 + r_2 \qquad \text{...(5.1)}$$

and $| f(z_1, z_2) |$ has to be calculated subject to the condition $r = r_1 + r_2$, $| z_1 | = r_1$ and $| z_2 | = r_2$. As z_1, z_2 vary over $\mathcal{S}^2$, it defines a norm on $X(\rho, d)$, the corresponding topology being stronger than $X(\rho, d, \delta)$. By using the method of Theorem 2, that the lattice product of normed topologies defined above on $X(\rho, d)$ as δ takes positive values is a complete linear metric space which will be stronger than $X(\rho, d)$. So by a well-known theorem of Banach (1932, Theorem 6, p, 41), the two metrics are equivalent.

Definition. Let z be a point in $\mathcal{S}^2$. Let $W(z_1, z_2)$ be a non-negative function defined on $\mathcal{S}^2$. Let $g_{mn}(z_1, z_2)$ be a double sequence of function and $g(z_1, z_2)$ be a function defined on $\mathcal{S}^2$. We say that $g_{mn}(z_1, z_2)$ converges to $g(z_1, z_2)$ with respect to $W(z_1, z_2)$, if for every $\epsilon > 0$, there exists $N(\epsilon)$ independent of z_1, z_2 in $\mathcal{S}^2$ such that

$$| g_{mn}(z_1, z_2) - g(z_1, z_2) | < \epsilon W(z_1, z_2) \text{ for all } m, n > N(\epsilon).$$

Using this definition, we have the following theorem, since the two topologies on $X(\rho, d)$ defined by (2.4) and (5.1) are equivalent.

Theorem 5 — Let $\{f_{mn}\}$ be a sequence in $X(\rho, d)$. Then $\{f_{mn}\} \to f$ in $X(\rho, d)$ is equivalent to the statement that for every $\delta > 0$, the sequence $f_{mn}(z_1, z_2)$ converges to the function $f(z_1, z_2)$ uniformly with respect to the function $\exp[(d + \delta) r^\rho]$ where $r = r_1 + r_2$, $| z_1 | = r_1$, $| z_2 | = r_2$.

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