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A STUDY OF THE LAGRANGIAN POINTS IN THE PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM

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In the present paper we have described a method for calculating the locations of Lagrangian points in photogravitational restricted problem and have investigated the stability of motion around these points.

INTRODUCTION

Radzievskii (1950) formulated the photogravitational restricted problem, which arises from the classical problem when one of the interacting masses is an intense emitter of radiation, and discussed it for three specific bodies : the Sun, a planet and a dust particle. In the present paper we have discussed a method for calculating the location of Lagrangian points and have investigated the stability of motion around these points.

EQUATIONS OF MOTION

The radiation repulsive force F_p exerted on a particle can be represented in terms of gravitational attraction F_g (Radzievskii 1950) as

$$F_p = F_g(1 - q). \qquad \ldots(1)$$

Here $q = 1 - (F_p/F_g)$, a constant for a given particle, is a reduction factor expressed in terms of the particle radius a, density δ and radiation-pressure efficiency factor x (in cgs system); as

$$q = 1 - \frac{5.6 \times 10^{-3}}{a\delta} x.$$

The assumption q = constant is equivalent to neglecting fluctuations in the beam of solar radiation and the effect of planet's shadow.

In dimensionless synodic co-ordinates (x, y), the equations of motion of the particle are

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$$\left.\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} \end{aligned}\right\} \quad \ldots(2)$$

where

$$\left.\begin{aligned} \Omega &= \tfrac{1}{2}\left[(1-\mu)\, r_1^2 + \mu r_2^2\right] + \frac{q(1-\mu)}{r_1} + \frac{\mu}{r_2} \\ r_1^2 &= (x-\mu)^2 + y^2; \quad r_2^2 = (x+1-\mu)^2 + y^2 \end{aligned}\right\} \quad \ldots(3)$$

and μ is the mass parameter of the system.

At libration points

$$\frac{\partial \Omega}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \Omega}{\partial y} = 0.$$

Or,

$$x - \frac{q(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3} = 0 \quad \ldots(4)$$

and

$$y\left[1 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right] = 0. \quad \ldots(5)$$

The solutions of eqns. (4) and (5) result in five points called the 'libration points' or the 'Lagrangian points', three of which are located on the x-axis and the other two are located symmetrically with respect to the x-axis.

Location of the Collinear Points

At the collinear libration points, $y = 0$ and their abscissa is given by eqn. (4). Denoting the expression of eqn. (4), after simplification, by $f(x)$, we observe that

$$f(-\infty) < 0; \; f(\mu - 1) > 0; \; f(\mu) < 0; \; f(\infty) > 0$$

so that the three collinear points $L_{1,2,3}$ lie in the intervals

$$-\infty \text{ and } \mu - 1; \; \mu - 1 \text{ and } \mu; \; \mu \text{ and } \infty,$$

respectively.

Now, as the first collinear point L_1 lines between $-\infty$ and $\mu - 1$ i.e., to the left of the smaller primary, we have, for this point

$$r_1 = \mu - x \quad \text{and} \quad r_2 = \mu - 1 - x.$$

Substituting these values of r_1 and r_2 in eqn. (4) and then putting $\xi_1 = \mu - 1 - x$, we get

$$\xi_1^5 + (3-\mu)\,\xi_1^4 + (3-2\mu)\,\xi_1^3 - \{\mu - (1-\mu)(1-q)\}\,\xi_1^2 - 2\mu\xi_1 - \mu = 0. \qquad \text{...(6)}$$

This is a fifth degree algebraic equation in ξ_1. Since there is only one change of sign in it, there exists at least one real root and as left hand side of this equation is less than zero for $\xi_1 = 0$ and infinite for $\xi_1 = \infty$, this root must be positive.

In a similar way, we have, for the second and third collinear points L_2 and L_3:

$$\xi_2^5 - (3-\mu)\,\xi_2^4 + (3-2\mu)\,\xi_2^3 - \{1 - q(1-\mu)\}\,\xi_2^2 + 2\mu\xi_2 - \mu = 0 \qquad \text{...(7)}$$

and

$$\xi_3^5 + (2+\mu)\,\xi_3^4 + (1+2\mu)\,\xi_3^3 - q(1-\mu)\,\xi_3^2 - 2q(1-\mu)\,\xi_3 - q(1-\mu) = 0 \qquad \text{...(8)}$$

where $\xi_2 = x + 1 - \mu$ and $\xi_3 = x - \mu$ respectively.

Thus, the three collinear libration points L_1, L_2, L_3 are located by

$$\left.\begin{aligned} x_1 &= \mu - 1 - \xi_1 \\ x_2 &= \mu - 1 + \xi_2 \\ \text{and} \quad x_3 &= \mu + \xi_3 \end{aligned}\right\} \qquad \text{...(9)}$$

where ξ_i $(i = 1, 2, 3)$ are given by eqns. (6), (7) and (8) respectively.

The variation in the values of x_i $(i = 1, 2, 3)$ for various values of the radiation factor q is shown in Figs. 1, 2 and 3 respectively. In Figs. 1–3, the curve for $q = 1.0$ corresponds to the classical case i.e., when solar radiation pressure is not taken into consideration, and the other two curves correspond to $q = 0.9$ and $q = 0.5$. It is observed that due to the radiation pressure the point L_1 is shifted towards the smaller primary whereas the points L_2 and L_3 are shifted towards the bigger primary. In other words, the tendency of radiation pressure is to bring the collinear points nearer to the source of radiation.

In may also be observed that $\triangle x_i$ $(i = 1, 2, 3)$, the deviation in the location of collinear points from the classical case, decreases with the increase in μ for the same value of q. Also, the deviation $\triangle x_i$ decreases with the increase in the value of q and this becomes zero for $q = 1.0$, which corresponds to the classical case.

Location of Triangular Points

For the triangular libration points, $y \neq 0$ and eqn. (5) gives

$$1 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3} = 0. \qquad \text{...(10)}$$

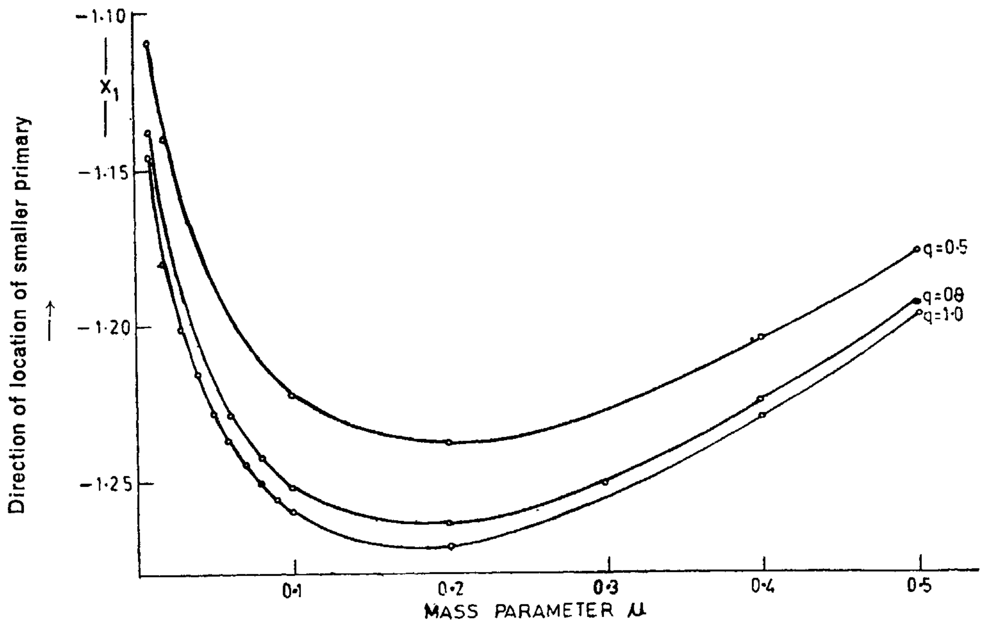

FIG. 1. Variation of location of first Lagrangian point L_1 with mass parameter μ and radiation factor q.

Rewriting (4) as

$$x\left[1 - \frac{q(1-\mu)}{r_1^3} - \frac{\mu}{r_2^3}\right] + \mu(1-\mu)\left[\frac{q}{r_1^3} - \frac{1}{r_2^3}\right] = 0$$

and making use of (10), we get

$$\frac{q}{r_1^3} - \frac{1}{r_2^3} = 0. \quad \text{...(11)}$$

From (10) and (11), we obtain

$$\left.\begin{aligned} r_1 &= (q)^{1/3} \leqslant 1 \\ r_2 &= 1. \end{aligned}\right\} \quad \text{...(12)}$$

On solving eqns. (12) for x and y, we get

$$\left.\begin{aligned} x_4 &= x_5 = \mu - \tfrac{1}{2}q^{2/3} = \mu - \tfrac{1}{2} + \tfrac{1}{3}p\,.(1 + \tfrac{1}{6}p) + O(p^3) \\ y_4 &= -y_5 = \tfrac{1}{2}q^{1/3}(4-q^{2/3})^{1/2} = \frac{\sqrt{3}}{2}\left(1 - \frac{2}{9}p - \frac{11}{81}p^2\right) + O(p^3) \end{aligned}\right\} \text{...(13)}$$

where $p = 1 - q$ is a small quantity. These eqns. (13) give the abscissae and the ordinates of the triangular libration points L_4 and L_5. On eliminating q from eqns. (13), we note that as q varies the points L_4 and L_5 lie on the curve

$$y^2 = 2(\mu - x) - (\mu - x)^2$$

or, $$x^2 + y^2 + 2(1 - \mu)\, x - (2 - \mu)\, \mu = 0$$

which is a unit circle with centre at $(\mu - 1, 0)$ i.e., at the smaller primary.

It may also be noted that in our problem the triangular libration points no longer form equilateral triangles with the primaries as they do in the classical case. Rather they form isoscles triangles with the primaries.

We observe that the ordinates of L_4 and L_5 are independent of mass parameter μ, but depend upon the radiation factor q and the abscissae of these points

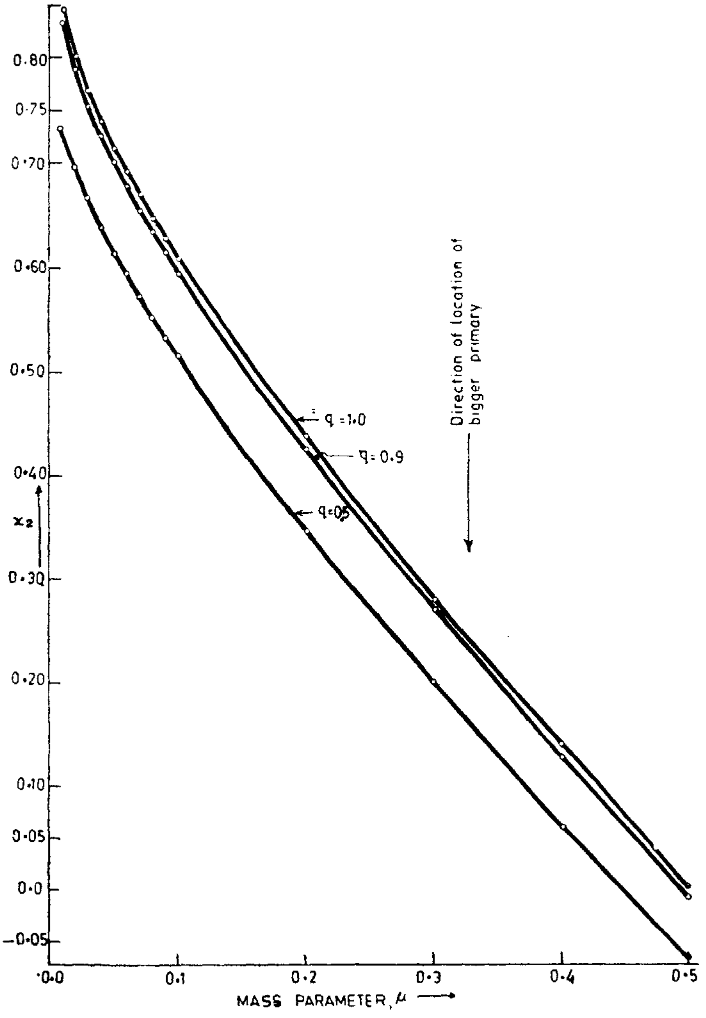

FIG. 2. Variation of location of 2nd Lagrangian point L_2 with mass parameter μ and radiation factor q.

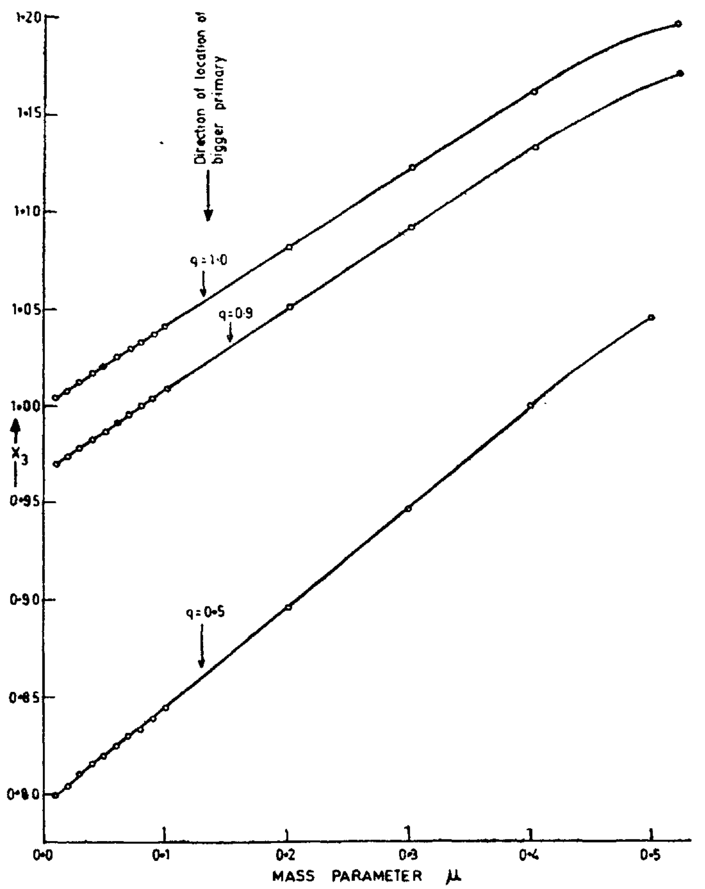

FIG. 3. Variation of location of 3rd Lagrangian point L_3 with mass parameter μ and radiation factor q.

depend both on μ and q. This is contrary to the classical case in which the abscissa is a function of μ only and the ordinate is a constant.

It is clear from eqns. (13) that for $0 \leqslant q \leqslant 2\sqrt{2}\mu^{3/2}$, we have $0 \leqslant x \leqslant \mu$ and for $2\sqrt{2}\mu^{3/2} \leqslant q \leqslant 1$, we have $0 \leqslant |x| \leqslant (\frac{1}{2} - \mu)$. This means that for values of $q < 2\sqrt{2}\mu^{3/2}$ the abscissa becomes positive which is never the situation in classical case. In the classical case, $x_4\ (= x_5)$ is either negative or at the most zero (for $\mu = 0.5$).

Also, second of eqns. (13) indicates that with decrease in the value of q (i.e., increase in the value of p), the libration point L_4 as well as L_5 move towards the line joining the two primaries and their symmetrical location with respect to this line is preserved.

STABILITY OF MOTION NEAR THE LIBRATION POINTS

For discussing the stability of motion near the libration point (x_0, y_0) we put

$$\xi = x - x_0$$

$$\eta = y - y_0$$

in the equations of motion and obtain the equations of variation:

$$\left.\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi\Omega^0_{xx} + \eta\Omega^0_{xy} \\ \ddot{\eta} + 2\dot{\xi} &= \xi\Omega^0_{xy} + \eta\Omega^0_{yy} \end{aligned}\right\} \qquad \text{...(14)}$$

retaining only linear terms in ξ and η. Here the superscript indicates that these partial derivatives of Ω are to be evaluated at the libration point (x_0, y_0).

First we consider the stability near the collinear points. We write

$$\left.\begin{aligned} \zeta_j &= \frac{q(1-\mu)}{(1 \pm \xi_j)^3} + \frac{\mu}{\xi_j^3} \; (j = 1, 2) \\ \zeta_3 &= \frac{q(1-\mu)}{\xi_3^3} + \frac{\mu}{(1 + \xi_3)^3} \end{aligned}\right\} \qquad \text{...(15)}$$

and find that

$$\left.\begin{aligned} \Omega^0_{xx}(L_i) &= 1 + 2\zeta_i \\ \Omega^0_{xy}(L_i) &= 0 \\ \Omega^0_{yy}(L_i) &= 1 - \zeta_i, \quad (i = 1, 2, 3) \end{aligned}\right\} \qquad \text{...(16)}$$

so that the characteristic equations at the collinear points L_i are

$$\lambda^4 + (2 - \zeta_i)\lambda^2 + (1 + 2\zeta_i)(1 - \zeta_i) = 0, \quad (i = 1, 2, 3). \qquad \text{...(17)}$$

Now, for $\mu < \frac{1}{2}$ and for corresponding values of ξ_i, it is found that

$$(1 + 2\zeta_i)(1 - \zeta_i) < 0 \text{ for every } i = 1, 2, 3$$

so that one root of (17), taken as a quadratic in λ^2, is always positive and another is negative. This indicates the presence of unbounded motion in the xy-plane, thereby making the points L_i $(i = 1, 2, 3)$ unstable.

In the case of triangular libration points, the characteristic equation corresponding to eqn. (14) is

$$\lambda^4 + \lambda^2 + \tfrac{9}{4}(4 - q^{2/3}) \cdot \mu \cdot (1 - \mu) = 0. \qquad \text{...(18)}$$

The solutions of eqn. (18) as a quadratic in $\Lambda = \lambda^2$ are

$$\Lambda_{1,2} = \tfrac{1}{2}\,[-1 \pm \{1 - 9(4 - q^{2/3}) \,.\, \mu \,.\, (1-\mu)\}^{1/2}] \qquad \text{...(19)}$$

Thus we see that the four roots $\lambda_{1,2,3,4}$ of the characteristic eqn. (18) depend not only upon the mass parameter μ but also on the radiation factor q. The nature of these roots is governed by the sign of the discriminant of eqn. (18). If d denotes this discriminant, then

$$d = 1 - \alpha\mu(1-\mu) \qquad \text{...(20)}$$

where

$$\alpha = 9(4 - q^{2/3}). \qquad \text{...(21)}$$

There are three possible cases regarding the sign of the discriminant d:

(I) When d is positive, the two roots in eqn. (20) are negative and consequently all the four roots of eqn. (18) are purely imaginary. This shows that the libration point in question is stable in the linear sense.

(II) When d is negative, the real part of the two characteristic roots are positive and equal so that the equilibrium point is unstable.

(III) For $d = 0$, the case of double roots arises which again indicates instability of the motion around the triangular points.

Critical Mass

The discriminant of the quadratic (18) is zero when

$$\alpha\mu^2 - \alpha\mu + 1 = 0.$$

Its solution for $0 \leqslant \mu \leqslant \frac{1}{2}$ is

$$\mu_{CSP} = \tfrac{1}{2}\left[1 - \left\{1 - \frac{4}{\alpha}\right\}^{1/2}\right] \qquad \text{...(22)}$$

$$= \mu_0 - \epsilon(p) \qquad \text{...(23)}$$

where $\epsilon(p) = 0.00891\,74706\;p - 0.00058\,15742\;p^2 + 0.00002\,94453\;p^3 + O(p^4)$. Here, μ_{CSP} is the critical value of the mass parameter μ in the present problem and $\mu_0(= 0.03852\,08965)$ is the critical value of μ when radiation pressure is not taken into consideration (Szebehely 1967). It may be noticed that μ_{CSP} is a function of q which is contrary to the classical case where critical mass is a constant quantity. In fact, in the classical case $p = 0$ so that $\mu_{CSP} = \mu_0$.

In Table I we have shown the values of the critical mass μ_{CSP} for various values of the radiation factor q, using the exact eqn. (22) and the approximate equation (up to third order of p) as given by (23). From the nature of the function

$\epsilon(p)$ it is noticed that with the inclusion of the radiation pressure from the bigger primary, the range of the mass parameter μ $(0 < \mu < \mu_0)$ which leads to stable motion around triangular points, decreases with q. Equation (23) gives a fairly good approximation for μ_{CSP} for values of q close to one which is generally the case in the solar system.

TABLE I

Radiation term q	μ_{CSP}, the critical value of mass parameter	
	Using eqn. (22)	Using eqn. (23)
1.0000	0.03852 08965	0.03852 08965
0.9999	0.03852 00048	0.03852 00048
0.9990	0.03851 19797	0.03851 19797
0.9900	0.03843 17795	0.03843 17799
0.9000	0.03763 44973	0.03763 49358
0.8000	0.03675 67658	0.03676 04297
0.7000	0.03588 41994	0.03589 72018
0.6000	0.03501 24007	0.03504 50752
0.5000	0.03413 55026	0.03420 38732

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