Indian J. pure appl. Math., **10**(7) : 785-791, July 1979

OSCILLATORY SHEARING IN HYPOELASTIC SOLID

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(*Received* 12 *July* 1978)

Oscillatory shearing of the type $x = X + f(t) \cos KZ$, $y = Y + f(t) \sin KZ$, $z = Z$ is studied for a hypoelastic layer $0 \leqslant Z \leqslant l$ of grade zero. It is found that oscillation is linear, the period of such oscillation has also been calculated. Stresses are obtained and it is observed that boundary should be maintained by normal force together with shearing force.

1. INTRODUCTION

Few dynamical problems have been studied in the field of hypoelastic material. Green studied a problem of simple extention (a dynamical theory) which is illustrated by Eringen (1965). Sinha (1961, 1963) considered torsional vibration and transverse waves in hypoelastic material. At present an oscillatory shearing of a hypoelastic layer bounded in the region $0 \leqslant Z \leqslant l$ has been studied. Carrol (1974) studied finite shearing oscillation where position of a practical at time t is $x = X + f(t) \cos KZ$, $y = Y + f(t) \sin KZ$, $z = Z$. Such type of deformation is considered in this paper for hypoelastic layer of grade zero and it is found that it is of linear type, $f(t)$ becomes a period function. It becomes interesting to note that such linear oscillation is possible for rubber-like Mooney-material in finite elastic theory as shown by Carrol (1974). All the stress-components are expressed. As in finite elasticity it is found that boundary should be maintained with a normal traction which is constant here along with shearing stress.

2. BASIC EQUATIONS

Basic equations of hypoelasticity of grade zero for homogeneous incompressible solid in absence of body forces are (Eringen 1965) :

Constitutive equation

$$2\mu \check{S}_i^k = - p\delta_i^k + 2\mu d_i^k \quad \text{...(2.1)}$$

Equation of motion

$$2\mu S^{ik}_{;i} = \rho \left(\frac{\partial v^k}{\partial t} + v^k_{;i} \, v^i \right) \quad \text{...(2.2)}$$

Conservation of mass

$$v^k_{;k} = 0 \qquad ...(2.3)$$

where

μ = shear modulus

δ^k_l = Kronecker's delta

ρ = density (constant)

S^k_l = stress-tensor

p = hydrostatic pressure

v^k = covariant velocity components

d^k_l = deformation rate (defined below)

$\check{S}^k_l$ = rate of stress-tensor (defined below)

$$d_{kl} = \tfrac{1}{2}(v_{k;l} + v_{l;k}) \qquad ...(2.4)$$

$$\check{S}^k_l = \frac{\partial S^k_l}{\partial t} + S^k_{l;m}\, v^m - S^{km} v_{l;m} + S^m_l\, v^k_{;m} \qquad ...(2.5)$$

(semi-colon indicates covariant differentiation).

3. Formulation of the Problem

Let us consider a layer of homogeneous incompressible hypoelastic material of grade zero in the region $0 \leqslant Z \leqslant l$, z-axis being vertically upward. In this medium a finite oscillatory shearing motion of the following type is considered :

$$x = X + f(t) \cos KZ,\ y = Y + f(t) \sin KZ,\ z = Z \qquad ...(3.1)$$

where x, y, z are the coordinates of a particle at time t and K is constant.

Velocity field corresponding to this motion is

$$v_1 = \dot{f} \cos Kz,\ \ v_2 = \dot{f} \sin Kz,\ \ v_3 = 0. \qquad ...(3.2)$$

Axes being rectangular, there will be no difference between covariant and contra-variant components.

Also let us assume that the hydrostatic pressure vanishes. Now it is our aim to study this type of shearing vibration.

4. Solution of the Problem

(a) *Period of Vibration*

Equation of conservation of mass is identically satisfied with (3.2). Equation of motion (2.2) gives

$$2\mu \frac{\partial S^{13}}{\partial z} = \rho \ddot{f} \cos Kz \qquad \text{...(4.1)}$$

$$2\mu \frac{\partial S^{23}}{\partial z} = \rho \ddot{f} \sin Kz \qquad \text{...(4.2)}$$

$$2\mu \frac{\partial S^{33}}{\partial z} = 0. \qquad \text{...(4.3)}$$

Rate of stresses (2.5) are calculated and are found to be

$$\check{S}^{11} = \frac{\partial S^{11}}{\partial t} + 2S^{13}\dot{f}K \sin Kz, \quad \check{S}^{22} = \frac{\partial S^{22}}{\partial t} - 2S^{23}\dot{f}K \cos Kz,$$

$$\check{S}^{33} = \frac{\partial S^{33}}{\partial t}, \quad \check{S}^{12} = \check{S}^{21} = \frac{\partial S^{12}}{\partial t} - S^{13}K\dot{f} \cos Kz + S^{32}K\dot{f} \sin Kz,$$

$$\check{S}^{31} = \check{S}^{13} = \frac{\partial S^{13}}{\partial t} + S^{33}K\dot{f} \sin Kz,$$

$$\check{S}^{23} = \check{S}^{32} = \frac{\partial S^{23}}{\partial t} - S^{33}K\dot{f} \cos Kz. \qquad \text{...(4.4)}$$

Rate of deformations are

$$d_{13} = -\frac{K}{2}\dot{f} \sin Kz, \quad d_{23} = \frac{K}{2}\dot{f} \cos Kz. \qquad \text{...(4.5)}$$

Other components are zero.

Now constitutive eqn. (2.1) gives

$$\frac{\partial S^{11}}{\partial t} + 2S^{13}\dot{f}K \sin Kz = 0 \qquad \text{...(4.6)}$$

$$\frac{\partial S^{22}}{\partial t} - 2S^{33}\dot{f}K \cos Kz = 0 \qquad \text{...(4.7)}$$

$$\frac{\partial S^{33}}{\partial t} = 0 \qquad \text{...(4.8)}$$

$$\frac{\partial S^{12}}{\partial t} - S^{13}K\dot{f} \cos Kz + S^{32}K\dot{f} \sin Kz = 0 \qquad \text{...(4.9)}$$

$$\frac{\partial S^{13}}{\partial t} + S^{33}K\dot{f} \sin Kz = -\frac{K}{2}\dot{f} \sin Kz \qquad \text{...(4.10)}$$

$$\frac{\partial S^{23}}{\partial t} - S^{33}K\dot{f}\cos Kz = \frac{K}{2}\dot{f}\cos Kz. \quad ...(4.11)$$

Equations (4.3) and (4.8) imply that S^{33} is independent of z and t i.e. a purely constant and is equal to the force per unit area applied at the bounding surface.

Differentiating either (4.10) [or (4.11)] w.r.t. z and then utilizing (4.1) [or (4.2)], we get

$$\frac{d^3f}{dt^3} = -\frac{(2S^{33}+1)}{\rho}\mu K^2\frac{df}{dt} \quad ...(4.12)$$

Its first integral is

$$\frac{d^2f}{dt^2} = -\frac{(2S^{33}+1)}{\rho}\mu K^2 f + A$$

where A is integration constant. In the position of centre of oscillation $f = 0$, $\frac{d^2f}{dt^2} = 0$ and hence $A = 0$.

Thus we get

$$\frac{d^2f}{dt^2} = -\frac{(2S^{33}+1)}{\rho}\mu K^2 f \quad ...(4.13)$$

which shows that vibration is periodic and period of vibration is

$$2\pi \Big/ \sqrt{\frac{(2S^{33}+1)\mu}{\rho}}K. \quad ...(4.14)$$

Therefore period depends inversely to the amount of shear.

Equation (4.13) may be integrated further to get

$$\dot{f}^2 = -\frac{(2S^{33}+1)}{\rho}\mu K^2 f^2 + B. \quad ...(4.15)$$

If $\dot{f}_0$ be the velocity at the centre of oscillation, then we get $\dot{f}_0^2 = B$. Now if f_m be the amplitude of vibration we will have relation between f_m and $\dot{f}_0$ as

$$\dot{f}_0^2 = \frac{(2S^{33}+1)}{\rho}\mu K^2 f_m. \quad ...(4.16)$$

If time be measured from the position of greatest displacement of particle from its centre of oscillation position, then solution of (4.13) will be

$$f = f_m \cos\sqrt{\frac{(2S^{33}+1)\mu}{\rho}}Kt. \quad ...(4.17)$$

(b) *Stress Distribution*

Integrating (4.1) and (4.2) we get

$$S^{13} = \frac{\rho}{2\mu K}\ddot{f}\sin Kz + g^{13}(t) \qquad ...(4.18)$$

$$S^{23} = -\frac{\rho}{2\mu K}\ddot{f}\cos Kz + g^{23}(t) \qquad ...(4.19)$$

where $g^{13}(t)$ and $g^{23}(t)$ are arbitrary functions of t.

Substituting (4.8) in (4.10) we get

$$\frac{\rho\dddot{f}}{2\mu K}\sin Kz + \dot{g}^{13}(t) + S^{33}K\dot{f}\sin Kz = -\frac{K}{2}\dot{f}\sin Kz.$$

Using (4.12) this becomes

$$-\frac{K}{2}(2S^{33}+1)\dot{f}\sin Kz + \dot{g}^{13} + S^{33}K\dot{f}\sin Kz = -\frac{K}{2}\dot{f}\sin Kz$$

i.e. $\dot{g}^{13}(t) = 0.$

$\therefore \quad g^{13}(t) =$ a purely constant quantity

$=$ initial value $= 0$

provided initial S^{13} is zero.

Similarly from (4.19) we will have

$g^{23}(t) =$ initial value $= 0$

on the assumption that initial S^{23} is zero.

Now substituting S^{13} in (4.6) we get

$$\frac{\partial S^{11}}{\partial t} = -2\dot{f}K\sin Kz\frac{\rho}{2\mu K}\ddot{f}\sin Kz = -\frac{\rho}{\mu}\sin^2 Kz\,\dot{f}\ddot{f}.$$

$$\therefore \quad S^{11} = -\frac{\rho}{2\mu}\sin^2 Kz\,\dot{f}^2 + g^{11}(z)$$

where $g^{11}(z)$ is an arbitrary function of z.

If initially $S^{11} = 0$, then we get $g^{11}(z) = 0$.

$$\therefore \quad S^{11} = -\frac{\rho}{2\mu}\sin^2 Kz\,\dot{f}^2.$$

Similarly substituting S^{23} in (4.7) and utilizing $S^{22} = 0$ initially we get

$$S^{22} = \frac{\rho}{2\mu}\cos^2 Kz\,\dot{f}^2.$$

Now from (4.9) on the use of S^{13} and S^{23} we get $\frac{\partial S^{12}}{\partial t} = 0$

i.e. S^{12} = initial value = 0, say.

Thus ultimately we get stresses as

$$S^{11} = -\frac{\rho}{2\mu} \sin^2 Kz\, \dot{f}^2, \quad S^{22} = \frac{\rho}{2\mu} \cos^2 Kz\, \dot{f}^2$$

S^{33} = force applied per unit area of the boundary = P, say

$$S^{13} = -\frac{(2P+1)}{2} K (\sin Kz)\, f \quad \text{[using 4.13]}$$

$$S^{23} = \frac{(2P+1)}{2} K (\cos Kz)\, f \quad \text{[using 4.13]}, \quad S^{12} = 0.$$

On the boundary $z = 0$ stresses are

$$S^{31} = 0, \; S^{32} = \frac{(2P+1)}{2} Kf, \quad S^{33} = P.$$

On the boundary $z = l$ stresses are

$$S^{31} = -\frac{(2P+1)}{2} Kf \sin Kl, \quad S^{32} = \frac{(2P+1)}{2} Kf \cos Kl, \quad S^{33} = P.$$

Thus the face $z = 0$ should be maintained except the normal force P, a tangential force along y-axis whereas the face $z = l$ is acted on by normal force P together with a tangential force having magnitude

$$\frac{(2P+1)}{2} Kf$$

making an angle $\left(\frac{\pi}{2} + Kl\right)$ with the positive sense of x-axis.

If K is choosen to be

$$K = (n + \tfrac{1}{2}) \frac{\pi}{l}, \quad n = 0, 1, 2, \ldots$$

tangential force on $z = l$ is along x-axis and thus we observe that in these cases oscillations in the faces $z = 0$ and $z = l$ are along x-axis and y-axis respectively.

5. Conclusion

From the above discussion we come to the following conclusions :

(i) A hypoelastic layer $0 \leqslant Z \leqslant l$ of grade zero may excecute a linear shearing oscillation having velocity field

$$u = \dot{f}(t) \cos KZ, \quad v = \dot{f}(t) \sin KZ, \quad \omega = 0.$$

It should be noted that such type of linearity in shearing oscillation holds for rubber-like Mooney-material in the field of homogeneous, incompressible finite elastic theory as shown in (3.9) of Carrol (1974).

(ii) For such shearing oscillation, the stress S^{33} is found to remain constant and so boundaries should be maintained by constant normal force together with shearing force. Similar analogy is also found in the case of simple shear problem of finite elastic theory (Eringen 1965).

Acknowledgement

The author remains grateful to Dr S. K. Chakraborty of Burdwan University for his kind help in the preparation of the paper.

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