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# ON THE ABSOLUTE NÖRLUND SUMMABILITY OF LAGUERRE SERIES

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In this paper, the authors prove a theorem on the absolute Nörlund summability of Laguerre series at the point $x = 0$, which improves the result of Yadav (1977).

## 1. INTRODUCTION

The Laguerre series associated with a function $f(x) \in L\,[0, \infty)$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) \qquad \ldots(1.1)$$

where

$$a_n = \{\Gamma(\alpha + 1)\, A_n^{\alpha}\}^{-1} \int_0^{\infty} e^{-x} x^{\alpha} f(x)\, L_n^{(\alpha)}(x)\, dx \qquad \ldots(1.2)$$

and $L_n^{(\alpha)}(x)$ denotes the $n$th Laguerre polynomial of order $\alpha > -1$.

Throughout the paper we write

$$\phi(y) = \{\Gamma(\alpha + 1)\}^{-1} [f(y) - f(0)]\, e^{-y} y^{\alpha}, \qquad \ldots(1.3)$$

$$F(y) = \{\Gamma(\alpha + 1)\}^{-1} [f(y) - f(0)]\, e^{-y} y^{(2\alpha-1)/4}. \qquad \ldots(1.4)$$

Recently, Yadav (1977) proved the following :

*Theorem A* — If $\{p_n\}$ be a non-negative and non-increasing sequence of constants such that $P_n = \sum_{k=0}^{n} p_k \to \infty$ and $\sum_n n^{(2\alpha-1)/4}/P_n$ is convergent, then the series (1.1) at the point $x = 0$ is $|N, p_n|$-summable, provided that $F(y)$ is bounded in $[0, \epsilon]$, $\epsilon > 0$, $\phi(y) \in BV\,[0, \infty)$ and

$$\int_n^{\infty} e^{-y/2} y^{(6\alpha-7)/12}\, |f(y)|\, dy = O(n^{-1/2}). \qquad \ldots(1.5)$$

We observe that in estimating $S_{n,2}$ he seems to have used

$$D_x L_n^{(\alpha)}(x) = O\left(x^{-(2\alpha-1)/4} \,.\, n^{(2\alpha-3)/4}\right)$$

while in the light of Szegö's relations (1959, p. 101 and p. 175) we see that

$$D_x L_n^{(\alpha)}(x) = - L_{n-1}^{(\alpha+1)}(x) = O(x^{-(2\alpha+3)/4} \cdot n^{(2\alpha+1)/4}) \qquad ...(1.6)$$

estimation of $S_{n.3}$ is also not proper. The asymptotic relation for $L_n^{(\alpha)}(x)$ in the interval $[\epsilon, n]$ is given in Szegö (1959) in some different form.

Considering the above facts, we prove the following theorem :

*Theorem* — If $\{p_n\}$ be a non-negative and non-increasing sequence of constants such that $P_n \to \infty$, and $\sum_n n^{(2\alpha+1)/4}/P_n$ is convergent, then the series (1.1) at the point $x = 0$ is $| N, p_n |$ -summable, provided that $F(y)$ is bounded in $[0, \epsilon]$, $\epsilon > 0$, $\phi(y) \in BV[0, \infty)$,

$$\int_\epsilon^n e^{y/2} y^{-(2\alpha+3)/4} \, | \phi(y) | \, dy = O(1) \qquad ...(1.7)$$

and

$$\int_n^\infty e^{y/2} y^{-(6\alpha+7)/12} \, | \phi(y) | \, dy = O(1), \text{ as } n \to \infty. \qquad ...(1.8)$$

## 2. Lemmas

For the proof of our theorem we shall use the following lemmas :

*Lemma* 1 (Szegö 1959, p. 175) — Let $\alpha$ be arbitrary and real, $c$ and $\epsilon$ fixed positive constants, and let $n \to \infty$. Then

$$L_n^{(\alpha)}(x) = \begin{cases} O(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}), \text{ if } \frac{c}{n} \leqq x \leqq \epsilon, & ...(2.1) \\ O(n^\alpha), \text{ if } 0 \leqq x \leqq \frac{c}{n}. & ...(2.2) \end{cases}$$

*Lemma* 2 Szegö 1959, p. 239) — If $\alpha$ and $\lambda$ be arbitrary and real, $\epsilon > 0$, $0 < \eta < 4$, then for $n \to \infty$

$$\max e^{-x/2} x^\lambda \, | L_n^{(\alpha)}(x) | \sim n^Q \qquad ...(2.3)$$

where

$$Q = \begin{cases} \max\left(\lambda - \tfrac{1}{2}, \frac{\alpha}{2} - \tfrac{1}{4}\right), \text{ if } \epsilon \leqq x \leqq (4 - \eta)\, n & ...(2.4) \\ \max\left(\lambda - \tfrac{1}{3}, \frac{\alpha}{2} - \tfrac{1}{4}\right), \text{ if } x \geqq \epsilon. & ...(2.5) \end{cases}$$

*Lemma* 3 (Bhatt 1962) — Let $\Sigma u_n$ be an infinite series with $S_n$ as its $n$th partial sum and $\{p_n\}$ be a positive non-increasing sequence such that $P_n \to \infty$. If $\sum_n |S_n|/P_n$ is convergent, then the series $\sum_n u_n$ is $|N, p_n|$-summable.

## 3. Proof of the Theorem

As given by Yadav (1977), we have

$$S_n(0) - f(0) = \int_0^\infty \phi(y)\, L_n^{(\alpha+1)}(y)\, dy$$

$$= \left\{ \int_0^{c/n} + \int_{c/n}^{\varepsilon} + \int_{\varepsilon}^{n} + \int_n^\infty \right\} \phi(y)\, L_n^{(\alpha+1)}(y)\, dy$$

$$S_{n\cdot1} + S_{n\cdot2} + S_{n\cdot3} + S_{n\cdot4}, \text{ say.}$$

Yadav (1977) has shown that

$$S_{n.1} = (n^{(2\alpha-1)/4}). \qquad \ldots(3.1)$$

Next,

$$S_{n.2} = \int_{c/n}^{\varepsilon} \phi(y) \frac{d}{dy} \{- L_{n+1}^{(\alpha)}(y)\}\, dy$$

$$= [\phi(y)\, L_n^{(\alpha+1)}(y)]_{c/n}^{\varepsilon} + \left| \int_{c/n}^{\varepsilon} d\{L_n^{(\alpha)}(y)\, \phi(y)\} \right|$$

$$+ \left| \int_{c/n}^{\varepsilon} \phi(y)\, dL_n^{(\alpha)}(y) \right|$$

$$= O\{n^{(2\alpha-1)/4} y^{(2\alpha-1)/4} \phi(y)\}_{c/n}^{\varepsilon} + O(n^{(2\alpha+1)/4}) \int_{c/n}^{\varepsilon} y^{-(2\alpha+3)/4} \phi(y)\, dy$$

$$= O\{n^{(2\alpha-1)/4} F(y)\}_{c/n}^{\varepsilon} + O(n^{(2\alpha+1)/4}) \int_{c/n}^{\varepsilon} F(y)\, y^{-1/2}\, dy$$

$$= O(n^{(2\alpha+1)/4}). \qquad \ldots(3.2)$$

Also, using (2.4) we get

$$S_{n\cdot3} = O(n^{(2\alpha+1)/4}) \int_{\varepsilon}^{n} e^{y/2} y^{-(2\alpha+3)/4} \,|\phi(y)|\, dy$$

$$= O(n^{(2\alpha+1)/4}). \qquad \ldots(3.3)$$

Finally, employing (2.5) and (1.8) we get

$$S_{n\cdot4} = O(n^{(2\alpha+1)/4}). \qquad \ldots(3.4)$$

Thus, by virtue of (3.1) ... (3.4), we find that

$$S_n = O(n^{(2\alpha+1)/4}).$$

Hence, by applying Lemma 3, we get the result contained in our Theorem.


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