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ON THE L^2 CLASSIFICATION OF A SECOND-ORDER MATRIX DIFFERENTIAL EQUATION

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We consider the second-order matrix differential equation $(N - \lambda)\phi = 0$, $(0 \leqslant x < \infty)$ and discuss the sufficient conditions on the coefficients under which the equation is not limit-2 at infinity.

§1. Let N denote the matrix differential operator

$$N \equiv \begin{pmatrix} -\frac{d}{dx}\left(p_0(x)\frac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & -\frac{d}{dx}\left(q_0(x)\frac{d}{dx}\right) + q_1(x) \end{pmatrix} \quad \ldots(1.1)$$

and ϕ a vector having two components $U \equiv U(x)$ and $V \equiv V(x)$ represented as a column matrix $\begin{pmatrix} U \\ V \end{pmatrix}$.

$p_0(x)$, $q_0(x)$, $p_1(x)$, $q_1(x)$ and $r(x)$ are all real-valued functions of x defined in $[0, \infty)$ satisfying

(i) $p_0(x)$, $q_0(x)$ are continuous and possess continuous derivatives on $[0, X]$ for all $X > 0$;

(ii) $p_0(x)$, $q_0(x) > 0$ for all $x \in [0, \infty)$;

(iii) $p_1(x)$, $q_1(x)$, $r(x) \in L\,[0, X]$ for all $X > 0$.

In an earlier paper the author (Bhagat 1969) has shown that for each λ such that $im\lambda \neq 0$, the matrix differential equation

$$(N - \lambda)\phi = 0 \quad (0 \leqslant x < \infty) \quad \ldots(1.2)$$

has at least two linearly independent solutions which belong to $L^2\,[0, \infty)$. Equation (1.2) may have three or all the four linearly independent solutions belonging to $L^2\,[0, \infty)$.

We say that the eqn. (1.2) is limit-2, limit-3 or limit-4 at infinity according as it has two, three or four linearly independent solutions belonging to $L^2\,[0, \infty)$. For similar classification of fourth order differential equations one may refer to Everitt (1962, 1963).

Everitt (1972) has discussed the limit-circle classification of second-order differential equation

$$My \equiv -(py')' + qy = \lambda y, \quad (0 \leqslant x < \infty)$$

and he has proved that, if

(a) p is absolutely continuous on $[0, X]$ for all $X > 0$;

(b) $q > -kx^{4/3}$ almost everywhere on $[0, \infty)$;

(c) $|p| < mx^{2/3}$ for all $x \in [0, \infty)$;

where k and m are non-negative constants, then the formally self-adjoint fourth-order differential equation

$$\psi^{(4)}(x) - (p(x)\,\psi^{(1)}(x))^{(1)} + (q(x) - \lambda)\,\psi(x) = 0 \quad (0 \leqslant x < \infty)$$

has exactly two linearly independent solutions, for each λ such that $im\lambda \neq 0$, which belong to $L^2\,[0, \infty)$.

Shaw and Bhagat (1974) have proved that under certain conditions to be satisfied by the coefficients the eqn. (1.2) has exactly two linearly independent solutions for each λ such that $im\lambda \neq 0$, which belong to $L^2\,[0, \lambda)$. One can also refer to Gadamsi and Mahto (1978) for another theorem on the limit-2 case.

In the present paper, we first state and prove sufficient conditions on the coefficients under which eqn. (1.2) is not limit-2 at infinity (§3). In §4 we discuss eqn. (1.2) in which the coefficients are of the form ax^{α}.

§2. Let D denote the subset of $L^2\,[0, \infty)$ of vectors defined by

$$y(x) = \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} \in D \text{ if}$$

(a) $y(x) \in L^2\,[0, \infty)$;

(b) $y'(x)$ is absolutely continuous in $[0, \infty)$;

(c) $Ny(x) \in L^2\,[0, \infty)$.

Let $[y_1 y_2](x)$ denote the bilinear concomitant of two vectors

$$y_1(x) = \begin{pmatrix} U_1(x) \\ V_1(x) \end{pmatrix} \text{ and } y_2(x) = \begin{pmatrix} U_2(x) \\ V_2(x) \end{pmatrix}$$

defined by (Shaw and Bhagat 1974, §2)

$$[y_1 y_2]\,(x) = p_0(x)\,(U_1^{(1)}\,\bar{U}_2 - U_1 \bar{U}_2^{(1)}) + q_0(x)\,(V_1^{(1)}\,\bar{V}_2 - V_1 \bar{V}_2^{(1)}). \quad \ldots(2.1)$$

Green's formula for these vectors is given by

$$\int_0^X (y_1^T N\bar{y}_2 - \bar{y}_2^T N y_1)\, dx = [y_1 y_2](X) - [y_1 y_2](0). \qquad ...(2.2)$$

From (2.2) it follows that $[y_1 y_2](X)$ tends to a finite limit as $X \to \infty$ for all $y_1, y_2 \in D$.

It follows from Shaw and Bhagat (1974, §2) that eqn. (1.2) has exactly two linearly independent solutions for each λ such that $im\lambda \neq 0$, which belong to $L^2[0, \infty)$, i.e. (1.2) is limit-2 at ∞, if and only if

$$\lim_{X \to \infty} [y_1 y_2](X) = 0 \qquad ...(2.3)$$

for all $y_1, y_2 \in D$.

§3. We now prove the following theorem :

Theorem — Let $p_0(x)$, $q_0(x)$, $p_1(x)$, $q_1(x)$, $r(x)$ in eqn. (1.2) satisfy the conditions (i), (ii) and (iii) of §1 and also satisfy the following conditions

(a) $p_0'(x)$, $q_0'(x)$, $p_1'(x)$, $q_1'(x)$ are absolutely continuous on $[0, X]$ for all $X > 0$;

(b) $p_0''(x)$, $q_0''(x)$, $p_1''(x)$, $q_1''(x) \in L^2[0, X]$ for all $X > 0$;

(c) $p_1(x)$, $q_1(x) < 0$ for all $x \in [0, \infty)$;

(d) $\begin{pmatrix} (-p_0 p_1)^{-1/4} \\ (-q_0 q_1)^{-1/4} \end{pmatrix}$ and $\begin{pmatrix} r(-p_0 p_1)^{-1/4} \\ r(-q_0 q_1)^{-1/4} \end{pmatrix} \in L^2[0, \infty)$;

(e) $\begin{pmatrix} \{p_0(p_0 p_1)' (-p_0 p_1)^{-5/4}\}' \\ \{q_0(q_0 q_1)' (-q_0 q_1)^{-5/4}\}' \end{pmatrix} \in L^2[0, \infty)$;

then eqn. (1.2) is not limit-2 at ∞.

In order to establish that (1.2) is not limit-2 at infinity it is sufficient to show that there are vectors y_1, $y_2 \in D$ such that

$$\lim_{X \to \infty} [y_1 y_2](X) \neq 0. \qquad ...(3.1)$$

Let us take $y_1(x) = y_2(x)$ and determine $y_1(x)$ by

$$y_1(x) = \begin{pmatrix} (-p_0(x)\, p_1(x))^{-1/4} \exp\left[i \int_0^x \left\{ -\frac{p_1(t)}{p_0(t)} \right\}^{1/2} dt \right] \\ (-q_0(x)\, q_1(x))^{-1/4} \exp\left[i \int_0^x \left\{ -\frac{q_1(t)}{q_0(t)} \right\}^{1/4} dt \right] \end{pmatrix}, \quad x \in [0, \infty). \qquad ...(3.2)$$

By actual calculation (details being omitted) it can be shown that

$$[y_1 y_2](x) = -4i \quad (x \in [0, \infty)), \qquad \ldots(3.3)$$

$y_1 \in L^2[0, \infty)$, by conditions (c) and (d),

y_1' is absolutely continuous in $[0, \infty)$, by condition (a),

and

$$\int_0^\infty |Ny_1|^2\, dx \leqslant \int_0^\infty [\{\tfrac{1}{4}(p_0(-p_0p_1)^{-5/4}(p_0p_1)')' + r(-q_0q_1)^{-1/4}\}^2$$

$$+ \{\tfrac{1}{4}(q_0(-q_0q_1)^{-5/4}(q_0q_1)')' + r(-p_0p_1)^{-1/4}\}^2]\, dx$$

$$< \infty, \text{ by conditions (d) and (e).}$$

Hence $\quad Ny_1 \in L^2[0, \infty)$.

Thus $\quad y_1 \in D$.

Now from (3.1) and (3.3) it follows that the differential eqn. (1.2) is not limit-2 at ∞.

§4. *Theorem* — In (1.1) let $p_0(x) = a_1x^{\alpha_1}$, $q_0(x) = a_2x^{\alpha_2}$, $p_1(x) = -b_1x^{\beta_1}$, $q_1(x) = -b_2x^{\beta_2}$ and $r(x) = cx^\gamma$ where $a_1, a_2, b_1, b_2, c, \alpha_1, \alpha_2, \beta_1, \beta_2$ and γ are real constants satisfying the conditions

(i) $a_1 \neq 0$, $a_2 \neq 0$: a_1 and b_1 are of the same sign and so are a_2 and b_2;

(ii) $\alpha_1 + \beta_1 > 2$, $\alpha_2 + \beta_2 > 2$;

(iii) $\alpha_1, \alpha_2 < 2$;

(iv) $4\gamma < \min(\alpha_1 + \beta_1 - 2, \alpha_2 + \beta_2 - 2)$;

then eqn. (1.2) is not limit-2 at ∞.

We shall define vectors $y_1(x)$ and $y_2(x)$ in D satisfying

$$\lim_{X \to \infty} [y_1 y_2](x) \neq 0.$$

Let us take

$$y_1(x) = y_2(x) = \begin{pmatrix} H_1(x) \exp[i \int_a^x S_1(t)\, dt] \\ \\ H_2(x) \exp[i \int_a^x S_2(t)\, dt] \end{pmatrix},$$

where

$$\left.\begin{aligned} H_1(x) &= (a_1 b_1 x^{\alpha_1+\beta_1})^{-1/4}, \quad H_2(x) = (a_2 b_2 x^{\alpha_2+\beta_2})^{-1/4} \\ S_1(x) &= \left(\frac{b_1}{a_1} x^{\beta_1-\alpha_1}\right)^{1/2}, \quad S_2(x) = \left(\frac{b_2}{a_2} x^{\beta_2-\alpha_2}\right)^{1/2}. \end{aligned}\right\} \qquad \ldots(4.1)$$

Then by actual calculation

$$[y_1 y_2](x) = -4i \quad (a \leqslant x < \infty), \qquad \ldots(4.2)$$

and

$$Ny_1 = \begin{pmatrix} (-a_1 x^{\alpha_1} H_1'(x))' \exp[i \int_a^x S_1(t)\,dt] + c x^{\gamma} H_2(x) \exp[i \int_a^x S_2(t)\,dt] \\ (-a_2 x^{\alpha_2} H_2'(x))' \exp[i \int_a^x S_2(t)\,dt] + c x^{\gamma} H_1(x) \exp[i \int_a^x S_1(t)\,dt] \end{pmatrix}. \qquad \ldots(4.3)$$

By condition (i), $H_1(x)$, $H_2(x)$, $S_1(x)$ and $S_2(x)$ are all real valued.

By condition (ii)

$$\begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix} \in L^2\,[a, \infty)$$

and so

$$y_1(x) \in L^2\,[a, \infty).$$

Now

$$H_1^{(r)}(x) = O(x^{-((\alpha_1+\beta_1)/4)-r}) \quad \text{and} \quad H_2^{(r)}(x) = O(x^{-((\alpha_2+\beta_2)/4)-r}), \quad 0 \leqslant r \leqslant 2.$$

Hence

$$\begin{aligned} \int_a^\infty |Ny_1|^2\,dx &\leqslant \int_a^\infty \{[(-a_1 x^{\alpha_1} H_1')' + c x^{\gamma} H_2]^2 \\ &\qquad + [(-a_2 x^{\alpha_2} H_2')' + c x^{\gamma} H_1]^2\}\,dx \\ &< \infty, \text{ by conditions (ii), (iii) and (iv).} \end{aligned}$$

Thus $\quad y_1 \in D$ and $\lim\limits_{x \to \infty} [y_1 y_2](x) \neq 0.$

This completes the proof of the theorem.

It has not been possible for the author to find sufficient conditions under which eqn. (1.2) is limit-3 or conditions under which it is limit-4.

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