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STRONG MEASURABILITY IN FRECHET SPACES

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Let $(X, \mathcal{U}, \mu)$ be a finite measure space, E a Frechet space, F a subspace of E' separating the points of E, and $f : X \to E$ such that $h \circ f$ is measurable for each $h \in F$. Some necessary and sufficient conditions are obtained for f to be weakly equivalent to a strongly measurable function.

In this paper $(X, \mathcal{U}, \mu)$ is a positive finite, complete, measure space, $\mathcal{U}$ being a σ-algebra of subsets of X, E a Hausdorff locally convex space over reals, and $f : X \to E$. f is said to be weakly measurable if $h \circ f$ is $\mathcal{U}$-measurable for every $h \in E'$, the topological dual of E. If E is a Frechet space, i.e., a complete metrizable locally convex space, then f is said to be strongly measurable if there exists a sequence $\{f_n\}$ of $\mathcal{U}$-simple functions from X into E such that $f_n \to f$, pointwise a.e. $[\mu]$. Two weakly measurable functions $f_i : X \to E$, $i = 1, 2$, are said to be weakly equivalent, written as $f \equiv g(w)$ if $\forall\, h \in E'$, $h \circ f = h \circ g$, a.e. $[\mu]$. If E is a Banach space, K is a weakly (i.e., $\sigma(E, E')$) compact subset of E, and $f : X \to K \subset E$, weakly measurable, then by a classical theorem of Phillips (1943), f is weakly equivalent to a strongly measurable function. [A new proof of this result is given by Tulcea (1973).] In this paper, using the theory of liftings (Tulcea and Tulcea 1969), we generalize the results of Edgar (1977) and Uhl (1978) to the case E is a Frechet space. For the measure space $(X, \mathcal{U}, \mu)$, we fix a lifting ρ and take the lifting topology $\mathcal{C}_\rho$ on X [Tulcea and Tulcea (1969, p. 59)]. For locally convex spaces, notations of Schaeffer (1971) and for measures on topological spaces, notations of Varadarajan (1965) are used.

Theorem 1 — Let E be a Hausdorff locally convex space, $f : X \to E$ be a weakly measurable function, and K a convex compact subset of $(E, \sigma(E, E'))$ such that for any $h \in E'$, $h \circ f \leqslant \sup\, h(K)$, a.e. $[\mu]$. Then there exists a weakly measurable function $g : X \to E$ such that $g \equiv f(w)$, $g(X) \subset K$, and $g : (X, \mathcal{C}_\rho) \to (E, \sigma(E, E'))$ is continuous.

PROOF : $(E, \sigma(E, E'))$ can be identified with a subspace of $R^{E'}$ with product topology. Thus f can be considered as $f : X \to R^{E'}$. For each $h \in E'$, let $g(h) = \rho(h \circ f)$ (note $h \circ f$ is essentially bounded) and let $g : X \to R^{E'}$, such that $(g)_h = g(h)$, $\forall\, h \in E'$. $g : (X, \mathcal{C}_\rho) \to R^{E'}$ is evidently continuous. Suppose $g(x_0) \notin K$ for some $x_0 \in X$. By the separation theorem (Schaeffer 1971, p. 65), there exists $h \in E'$ such that $h \circ g(x_0) > \sup h(K)$. Now $h \circ f \leqslant \sup h(K)$, a.e. $[\mu]$, implies $\rho(h \circ f) \leqslant \sup h(K)$ everywhere and so $h \circ g \leqslant \sup h(K)$ everywhere, a contradiction. This proves the theorem.

Theorem 2 — Let E be a Frechet space, F, a subspace of E', separating the points of E, and $f : X \to (E, \sigma(E, F))$ weakly measurable (this means $h \circ f$ is $\mathcal{U}$-measurable for each $h \in F$). Then there exists a strongly measurable function $g : X \to E$ such that $h \circ g = h \circ f$ a.e. $[\mu]$, $\forall\, h \in F$, if and only if there exists a sequence $\{K_i\}_{1 \leqslant i < \infty}$ of $\sigma(E, E')$ compact convex subsets of E and a $\mathcal{U}$-partition of $X = \bigcup_{i=0}^{\infty} X_i$ such that $\mu(X_0) = 0$, $\mu(X_i) > 0$, $\forall\, i \geqslant 1$, and for every i, on X_i, $h \circ f \leqslant \sup h(K_i)$, a.e. $[\mu]$, for each $h \in F$.

Proof : If there is a strongly measurable $g : X \to E$, such that $h \circ g = h \circ f$, a.e. $[\mu]$, for each $h \in F$, then by Egorov's theorem (Dincleanu 1967, p. 94) it is easily seen that the conditions of the theorem are satisfied. Conversely suppose first there is a $\sigma(E, E')$-compact convex subset K of E such that $h \circ f \leqslant \sup h(K)$, a.e. $[\mu]$, for each $h \in F$. By Theorem 1, there is a weakly measurable $f_0 : X \to (E, \sigma(E, F))$ such that $f \equiv f_0(w)$, and $f_0(X) \subset K$. Let $\alpha_i = \sup p_i(K_0)$, K_0 being the closed absolutely convex hull of K_0 (note K_0 is weakly compact) and $\{p_i\}$ being an increasing sequence of seminorms generating the topology of E. On $E_0 = \bigcup_{n=1}^{\infty} nK_0$, we define a norm $\|\cdot\| = \sum_{n=1}^{\infty} (1/2^n \alpha_n)\, p_n$. It is a simple verification that the topologies induced on K_0 by E and the normed space $(E_0, \|\ \|)$ are identical (cf. Saab 1975) and K_0 is weakly compact in $(E_0, \|\cdot\|)$ and so is also weakly compact in the Banach space $(E_1, \|\cdot\|)$, the completion of $(E_0, \|\cdot\|)$. Let P be the sublattice generated by $F \mid_{K_0}$ in $C(K_0)$, the lattice of all continuous real-valued functions on K with weak-topology (i.e., one induced by $\sigma(E, E')$). Since $h \circ f_0$ is measurale for every $h \in F$, $l \circ f$ is measurable for every $l \in P$. By the Stone-Weirestrass approximation theorem, $l \circ f_0$ is measurable for every $l \in C(K_0)$. Thus $f_0 : X \to E_1$ is weakly measurable and so by Phillips' theorem, there exists a strongly measurable $g : X \to E_1$, $g(X) \subset K$, such that $g \equiv f_0(w)$ (note anything here is with respect to the Banach space E_1). Thus the result is proved in this particular case. The general case can now be easily handled since we get, $\forall\, i$, $1 \leqslant i < \infty$, by the above argument a strongly measurable function $g_i : X_i \to K_i$ such that $h \circ g_i = h \circ (f \mid_{X_i})$, a.e. $[\mu]$, $\forall\, i \geqslant 0$.

Corollary 3 — Let E be a Frechet space, $F \subset E'$, a separating subspace of E' (i.e. F separates the points of E), and $f : X \to E$ such that $h \circ f$ is measurable, $\forall\, h \in F$. Then there exists a strongly measurable $g : X \to E$ such that $h \circ g = h \circ f$, a.e. $[\mu]$, $\forall\, h \in F$, if and only if for any $A \in \mathcal{U}$ with $\mu(A) > 0$, there exists a

$B \in \mathcal{U}$ with $B \subset A$ and $\mu(B) > 0$ and a weakly relatively compact subset K of E such that, on B, $h \circ f \leqslant \sup h(K)$, a.e. $[\mu]$, for every $h \in F$.

PROOF : If there is a strongly measurable $g : X \to E$ such that $h \circ f = h \circ g$, a.e. $[\mu]$, $\forall\, h \in F$, then the result follows from Theorem 2. If the conditions of the theorem are satisfied, by Zorn's Lemma we get a $\mathcal{U}$-partition of $X = \bigcup_{i=0}^{\infty} X_i$, with $\mu(X_0) = 0$ and $\mu(X_i) > 0$ for each $i > 0$, and a sequence $\{K_i\}_{1 \leqslant i < \infty}$ of weakly compact convex subsets of E, such that for every $i \geqslant 1$, $h \circ f \mid X_i \leqslant \sup h(K_i)$, a.e. $[\mu]$, for every $h \in F$. Theorem 2 now gives the result.

Remark 4: Theorem 1 and Corollary 2 of Uhl (1978) are particular cases of this corollary.

It has been proved by Edgar (1977) that if E is a Hausdorff locally convex space and $f : X \to E$ is weakly measurable, then the image measure $\nu : \mathcal{B} \to R$, $\nu(B) = \mu(f^{-1}(B))$, is a Baire measure on $(E, \sigma(E, E'))$, $\mathcal{B}$ being the class of all Baire subsets of $(E, \sigma(E, E'))$ (Edgar 1977, Varadarajan 1965). It follows from this that if $f_i : X \to E$, $i = 1, 2$, are weakly measurable and $f_1 \equiv f_2(w)$, then their image measures will be identical. With these notations we prove the following.

Corollary 5 — Let E be a Frechet space, F, a subspace of E', separating the points of E, and $f : X \to (E, \sigma(E, F))$ weakly measurable. Then there exists a strongly measurable $g : X \to E$ such that $h \circ g = h \circ f$, a.e. $[\mu]$, $\forall\, h \in F$ if and only if the image Baire measure ν on $(E, \sigma(E, F))$ is inner regular by $\sigma(E, E')$-compact subsets of E.

PROOF : Suppose there exists a strongly measurable function $g : X \to E$ such that $h \circ g = h \circ f$, a.e. $[\mu]$, $\forall\, h \in F$. By Theorem 2 there is a $\mathcal{U}$-partition $X = \bigcup_{i=0}^{\infty} X_i$ and a sequence $\{K_i\}$ of compact convex subsets of $(E, \sigma(E, E'))$ such that $\mu(X_0) = 0$, $\mu(X_i) > 0$, and on X_i, $h \circ f \leqslant \sup h(K_i)$, a.e. $[\mu]$, $\forall\, h \in F$, for each i. By Theorem 1, there exists a weakly measurable $\gamma_i : X_i \to (E, \sigma(E, F))$ such that $\gamma_i(X_i) \subset K_i$ and $\gamma_i \equiv f \mid X_i\, (w)$. Thus we get

$$f_0 : X \to E,\; f_0(x) = \begin{Bmatrix} \gamma_i(x),\; x \in X_i,\; i \geqslant 1 \\ 0 \;,\; x \in X_0 \end{Bmatrix},\; f \equiv f_0(w)$$

and $f_0(X_i) \subset K_i$. Let $\nu : \mathcal{B} \to R^+$ be the image measure of f or f_0. Choose an n such that $\mu(\bigcup_{i>n} X_i) < \epsilon$. Take any $B \in \mathcal{B}$, $B \supset \bigcup_{i=1} K_i$. Thus

$$\nu(B) \geqslant \mu(\bigcup_{i=0}^{n} X_i) \geqslant \mu(X) - \epsilon.$$

This proves the inner regularity of ν by compact subsets of $(E, \sigma(E, E'))$. Conversely if ν is inner regular by the compact subsets of $(E, \sigma(E, E'))$, we will use Corollary 3 to prove that there exists a strongly measurable $g : X \to E$ such that $h \circ g = h \circ f$, a.e. $[\mu]$, $\forall\ h \in F$. Take an $A \in \mathcal{U}$ with $\mu(A) > \eta > 0$ and let K be a compact subset of $(E, \sigma(E, E'))$ such that $\inf \{\mu(f^{-1}(C)) : C \in \mathcal{B},\ C \supset K\} > \mu(X) - \eta/2$. Take a $C_0 \in \mathcal{B}$, $C_0 \supset K$, such that $\inf \{\mu(f^{-1}(C)) : C \in \mathcal{B}, C \supset K\} = \mu(f^{-1}(C_0))$. Take $h \in E'$ and let $P = \{y \in C_0 : h(y) > \sup h(K)\}$. Then $P \in \mathcal{B}$ and $P \cap K = \phi$. Thus $\mu(f^{-1}(P)) = 0$. Put $B = A \cap f^{-1}(C_0)$, we see $\mu(B) > \eta/2 > 0$, and $h \circ f \leqslant \sup h(K)$, on B, a.e. $[\mu]$. The result now follows from Corollary 3.

In the next theorem we generalize Theorem 3 of Tulcea (1974).

Theorem 6 — Let E be a metrizable locally convex space and $f : X \to E$ weakly measurable. Then f is strongly measurable if and only if there is an $X_0 \in \mathcal{U}$ with $\mu(X_0) = \mu(X)$, such that for any $h_1 \in E'$ and $h_2 \in E'$, $h_1 \circ f \mid X_0 \neq h_2 \circ f \mid X_0$ implies $h_1 \circ f \neq h_2 \circ f$ a.e. $[\mu]$.

PROOF : Let $\{p_n\}$ be an increasing sequence of seminorms generating the topology of E. If f is strongly measurable, there exists a sequence $f_n : X \to E$ of $\mathcal{U}$-simple functions, which we can assume to be continuous, with $\mathcal{C}_\rho$ topology on X, such that $f_n \to f$, pointwise a.e. $[\mu]$. By Ergorov's theorem (Dincleanu 1967, p. 94) there is an increasing sequence $\{X_k\}_{k\geqslant 1}$, $X_k \in \mathcal{U}$, $\mu(X_k) > 0$, $\mu(X_k) > \mu(X) - (1/k)$, and $f_n \to f$ uniformly on X_k, $\forall\ k$. Put $X_0 = \bigcup_{k=1}^{\infty} ((\rho(X_k)) \cap X_k)$. Then $\mu(X_0) = \mu(X)$. Take an $h \in E'$ such that $h \circ f = 0$ a.e. $[\mu]$. Assume there is an

$$x \in X_k \cap (\rho(X_k)) = Y_k, \text{ say,}$$

for some k such that $h \circ f(x) \neq 0$. Since $f_n \to f$ uniformly on Y_k, $f \mid Y_k$ is continuous, the topology being the one induced by $\mathcal{C}_\rho$. Thus there is an $H \in \mathcal{U}$, with $\rho(H) = H$ and $x \in H$, such that $\mid h \circ f(H \cap Y_k) \mid > 0$. From

$$\rho(H \cap Y_k) = \rho(H) \cap \rho(X_k) = H \cap \rho(X_k)$$

it follows that $\mu(H \cap Y_k) > 0$ and so $h \circ f \neq 0$ a.e. $[\mu]$.

Now we shall prove the converse. For notational convenience we assume $X = X_0$ (otherwise restrict everything to X_0). For each n, we get the normed space $E_n = (E/p_n^{-1}\{0\}, p_n)$ whose completion we denote by $\tilde{E}_n$. The mapping f gives rise to the mappings $f_n : X \to E_n$, $f_n = \varphi_n \circ f$, $\varphi_n : E \to E_n$ being the canonical mapping. By hypothesis and Theorem 3 of Tulcea (1974) $f_n : X \to \tilde{E}_n$ are strongly measurable. Thus $f_n : X \to E_n$ are strongly measurable for every n. Since E can be embedded in $\Pi\, E_n$ (Dincleanu 1967, Theorem 3, p. 100 and Theorem 2, p. 99), f is strongly measurable.

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