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ESTIMATION PROBLEMS IN BIVARIATE LOGNORMAL DISTRIBUTION

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By considering the regressions equations of the bivariate lognormal distribution, we have been able to estimate the variance regression of one variate given the other. Unbiased and asymptotically unbiased estimates are obtained. We have also considered other estimates of this regression and their efficiencies have been studied.

1. INTRODUCTION

The problem of estimating the parameters of the lognormal population has been considered by many authors.

Finney (1941) has considered the problem of estimating the mean and the variance of the population from a sample when it is known that the logarithm of an observation is normally distributed. He obtained efficient estimates of the mean and of the variance population.

Mostafa and Mahmoud (1966) considered the problem of estimating the mean, the median, the mode, the variance of the univariate lognormal population. They suggested estimates for the mean and the variance of the population other than those suggested by Finney (1941). These estimates in despite of being biased are easier for numerical calculation. Also estimates for the median and the mean have been suggested.

Mostafa and Mahmoud (1964) considered the problem of estimating the mean, median and modal regression of one variate given the other for the bivariate lognormal population. Unbiased estimates have been obtained together with other estimates. The efficiency of the estimates has been studied.

In this paper we estimate the variance regression of one variate given the other for the bivariate lognormal population. Unbiased estimate and asymptotically unbiased estimates have been suggested. The efficiency of these estimates have been studied.

2. THE VARIANCE REGRESSION

Given the two variates $X_1 : N(\alpha_1, \sigma_1^2)$, $X_2 : N(\alpha_2, \sigma_2^2)$ and $X_1 = \log z_1$, $X_2 = \log z_2$. Assuming that their joint distribution is bivariate normal then the joint distribution of z_1, z_2 is

$$\phi(z_1, z_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \cdot \frac{1}{z_1 z_2} \exp\left(-\frac{1}{2(1-\rho^2)}(A^2 - 2\rho AB + B^2)\right) \quad \ldots(2.1)$$

where

$$A = (\log z_1 - \alpha_1)/\sigma_1 \quad \text{and} \quad B = (\log z_2 - \alpha_2)/\sigma_2.$$

The conditional distribution of z_2 given z_1 is

$$\phi(z_1/z_2) = \frac{1}{\sqrt{2\pi}\,\sigma z_2} \exp\left(-\tfrac{1}{2}\left(\frac{\log z_2 - m}{\sigma}\right)^2\right) \quad \ldots(2.2)$$

where

$$m = \alpha_2 + \rho\frac{\sigma_2}{\sigma_1}(\log z_1 - \alpha_1), \quad \sigma^2 = \sigma_2^2(1-\rho^2).$$

Thus it can be shown that

$$E((z_2^t/z_1)) = \exp(tm + \tfrac{1}{2}\sigma^2 t^2) \quad \ldots(2.3)$$

$$\text{var}(z_2/z_1) = \exp\left(2\left(\left(\alpha_2 + \rho\frac{\sigma_2}{\sigma_1}(\log z_1 - \alpha_1)\right) + \sigma_2^2(1-\rho^2)\right)\right.$$
$$\left.(\exp(\sigma_2^2(1-\rho^2)) - 1\right) = e^{2m}(e^{2\sigma^2} - e^{\sigma^2}) \quad \ldots(2.4)$$

which can be written as

$$\text{var}\left(\frac{z_2}{z_1}\right) = \left(\exp\left(2E\left(\frac{X_2}{X_1}\right) + \text{var}\left(\frac{X_2}{X_1}\right)\right)\right)\left(\exp\text{var}\left(\frac{X_2}{X_1}\right)\right) - 1\Big). \quad \ldots(2.5)$$

3. Information Required for the Problem

When X_1, X_2 have joint normal distribution it is well known that

(i) $E(X_2/X_1) = \alpha_2 + \beta_2(X_1 - \alpha_1)$

where

$$\alpha_2 = E(X_2) \quad \text{and} \quad \beta_2 = \rho\frac{\sigma_2}{\sigma_1}. \quad \ldots(3.1)$$

(ii) The (mean) regression equation of X_2 given X_1 (based on n pairs of observations) is

$$X_2 = \hat{\alpha}_2 + \hat{\beta}_2(X_1 - \bar{X}_1) + e \quad \ldots(3.2)$$

where

$$\hat{\alpha}_2 = \bar{X}_2$$

$$\hat{\beta}_2 = \Sigma (X_1 - \bar{X}_1)(X_2 - \bar{X}_2)/(X_1 - \bar{X}_1)^2 = \Sigma C_i X_{2i}$$

and e is a random error.

It can be shown that (Mostafa and Mahmoud 1964)

$\hat{\xi} = \hat{\alpha}_2 + \hat{\beta}_2(X_1 - \bar{X}_1)$ is normally distributed with

$$E\left(\frac{\hat{\xi}}{X_1}\right) = \alpha_2 + \beta_2(X_1 - \alpha_1) = m \qquad \text{...(3.3)}$$

and

$$\text{var}\left(\frac{\hat{\xi}}{X_1}\right) = \frac{\sigma^2}{n} + \frac{(X_1 - \bar{X}_1)^2\sigma^2}{\Sigma (X_1 - X_1)^2} = U^2. \qquad \text{...(3.4)}$$

Putting $K_n = (X_1 - \bar{X}_1)^2/\Sigma (X_1 - \bar{X}_1)^2$,

$K_n = K_n(X_1; X_{11}, X_{12}, X_{13}, \ldots, X_{1n})$.

It is clear that K_n is monotonic decreasing and bounded,

$K_{n+1} < K_n$, for all n

$0 < K_n < 1$.

As $n \to \infty$, $\lim K_n = K$, where $0 < K < 1$.

Then

$$U^2 = (1 + nK_n)\,\sigma^2/n.$$

Hence

$$E(\exp(r\hat{\xi})) = \exp(rm + \tfrac{1}{2} r^2 U^2). \qquad \text{...(3.5)}$$

In particular

$$E(\exp(\hat{\xi})) = \exp(\alpha_2 + \beta_2(X_1 - \alpha_1) + \tfrac{1}{2}(1 + nK_n)\,\sigma^2 n). \qquad \text{...(3.6)}$$

Also if

$$\zeta = \frac{1}{2(n-2)} F'F$$

where $F'F/\sigma^2$ is the residual sum of squares, then it can be shown that (Mostafa and Mahmoud 1964)

$$E(\zeta^i e^{j\zeta}) = \frac{\sigma^{2i}}{(n-2)^i} \cdot \frac{\Gamma((i + \frac{1}{2}(n-2))}{\Gamma((\frac{1}{2}(n-2))} \left(1 - \frac{j\sigma^2}{n-2}\right)^{-(2i+n-2)/2}. \qquad \text{...(3.7)}$$

In particular if $i = r$, $j = 0$, we get

$$E(\zeta^r) = \frac{\sigma^{2r}}{(n-2)^r} \cdot \frac{\Gamma(r + \frac{1}{2}(n-2))}{\Gamma(\frac{1}{2}(n-2))} \qquad \text{...(3.8)}$$

and if $i = 0$, $j = r$, we get

$$E(e^{r\zeta}) = \left(1 - \frac{r\sigma^2}{n-2}\right)^{-(n-2)/2} \quad \text{for } r < \frac{n-2}{\sigma^2}.$$

4. Estimation of the Variance Regression of z_2 given z_1

A natural estimate N for var (z_2/z_1) can be obtained by replacing in eqn. (2.4) the population values by their values from the sample. Thus we get

$$N = e^{2\hat{\xi}}(e^{4\zeta} - e^{2\zeta}). \qquad \ldots(4.1)$$

From (3.5) and (3.7) we get

$$E(N) = \left(\exp\left(2m + 2\sigma^2\left(K_n + \frac{1}{n}\right)\right)\right)\left(\left(1 - \frac{4\sigma^2}{n-2}\right)^{-(n-2)/2} - \left(1 - \frac{2\sigma^2}{n-2}\right)^{-(n-2)/2}\right), \qquad \ldots(4.2)$$

i.e. N is biased estimate for var (z_2/z_1).

When

$$n \to \infty, \; E(N) \to \exp(2m + 2\sigma^2 K)\,(e^{2\sigma^2} - e^{\sigma^2}). \qquad \ldots(4.3)$$

This shows that the estimate is biased even when $n \to \infty$, but

when $n \to \infty$, $K \to 0$, we have $E(N) \to e^{2m}(e^{2\sigma^2} - e^{\sigma^2})$.

Also when $\rho = 0$, (4.2) becomes

$$E(N) = \left(\exp\left(2\alpha_2 + 2\sigma_2^2\left(K_n + \frac{1}{n}\right)\right)\right)\left(\left(1 - \frac{4\sigma_2^2}{n-2}\right)^{-(n-2)/2} - \left(1 - \frac{2\sigma_2^2}{n-2}\right)^{-(n-2)/2}\right) \qquad \ldots(4.4)$$

and (4.3) becomes

$$E(N) \to \exp(2\alpha_2 + 2\sigma_2^2 K)\,(e^{2\sigma_2^2} - e^{\sigma_2^2}). \qquad \ldots(4.5)$$

Equation (4.4) and (4.5) mean that the estimate N shows dependence when there is really independence even when $n \to \infty$.

The expected value of N to the order n^{-1} is

$$E(N) = \exp(2m + 2K_n\sigma^2)\left(\left(1 + \frac{2\sigma^2}{n} + 4\,\frac{\sigma^4}{n}\right)e^{2\sigma^2} - \left(1 + \frac{2\sigma^2}{n} + \frac{\sigma^4}{n}\right)e^{\sigma^2}\right). \qquad \ldots(4.6)$$

Hence the percentage of the biasedness to the order n^{-1} is

$$100 \times \frac{E(N) - \text{var}\,(z_2/z_1)}{\text{var}\,(z_2/z_1)} = \frac{((1 + 2\sigma^2/n + 4\sigma^4/n) \exp(2K_n\sigma^2) - 1)\, e^{\sigma^2}}{(e^{\sigma^2} - 1)} - \frac{((1 + 2\sigma^2/n + \sigma^4/n) \exp(2K_n\sigma^2) - 1)}{(e^{\sigma^2} - 1)} \times 100.$$

The variance of the estimate N to order n^{-1} is given by

$$\begin{aligned}\text{var}\,(N) = {} & e^{4m+2\sigma^2}\left(\left(1 + \frac{8\sigma^2}{n} + \frac{16\sigma^2}{n}\right)\exp(8K_n\sigma^2) - \left(1 + \frac{4\sigma^2}{n} + \frac{8\sigma^4}{n}\right)\exp(4K_n\sigma^2)\right) \\ & + e^{4m+2\sigma^2}\left(\left(1 + \frac{8\sigma^2}{n} + \frac{4\sigma^4}{n}\right)\exp(8K_n\sigma^2) - \left(1 + \frac{4\sigma^2}{n} + \frac{2\sigma^4}{n}\right)\exp(4K_n\sigma^2)\right) \\ & - 2e^{4m+3\sigma^2}\left(\left(1 + \frac{8\sigma^2}{n} + \frac{9\sigma^4}{n}\right)\exp(8K_n\sigma^2) - \left(1 + \frac{4\sigma^2}{n} + \frac{5\sigma^4}{n}\right)\exp(4K_n\sigma^2)\right).\end{aligned}$$

Also we have as $n \to \infty$, var $(N) \to e^{4m}(e^{8K\sigma^2} - e^{4K\sigma^2})\,(e^{2\sigma^2} - e^{\sigma^2})^2$ but if $K \to 0$, var $(N) \to 0$.

Now it is clear that the estimate N is a poor estimate for var (z_2/z_1) and for this reason we can estimate var (z_2/z_1) by estimate of the form

$$M_i = e^{2\hat{\xi}} g_i(\zeta) \qquad \ldots(4.7)$$

where $\hat{\xi} = \hat{\alpha}_2 + \hat{\beta}_2(X_1 - \bar{X}_1)$, $\zeta = \frac{1}{2}\hat{\sigma}^2$.

We attempt to choose $g_i(\zeta)$ such that M_i is unbiased i.e.

$$E(M_i) = e^{2m}(e^{2\sigma^2} - e^{\sigma^2}). \qquad \ldots(4.8)$$

Since ζ and $\hat{\xi}$ are independent then

$$E(g_i(\zeta)) = E(M_i)\,(Ee^{2\hat{\xi}})^{-1}. \qquad \ldots(4.9)$$

From (4.8) and (4.9) we get

$$E(g_i(\zeta)) = \exp\left(2\sigma^2\left(1 - \left(\frac{K_n + 1}{n}\right)\right)\right) - \exp\left(\sigma^2\left(1 - 2\left(\frac{K_n + 1}{n}\right)\right)\right). \qquad \ldots(4.10)$$

Putting $\lambda_n = (1 - (K_n + 1/n))$,

$$\lambda_n = \lambda_n(X_1; X_{11}, X_{12}, \ldots, X_{1n}).$$

It is clear that as $n \to \infty$ $\lambda_n \to 1 - K$.

$$E(g_i(\zeta)) = e^{2\lambda_n \sigma^2}(1 - e^{-\sigma^2}). \quad \ldots(4.11)$$

To obtain $g_i(\zeta)$, we suggest the following two approaches.

(I) In the first approach we insert in place of σ^2 its estimates 2ζ and take $g_1(\zeta)$ as

$$g_1(\zeta) = e^{4\lambda_n \zeta}(1 - e^{-2\zeta}). \quad \ldots(4.12)$$

Hence we get the estimate M_1 for var (z_2/z_1) defined by

$$M_1 = \exp(2\xi + 4\lambda_n \zeta)(1 - e^{2\zeta}). \quad \ldots(4.13)$$

The expected value of M_1 to the order n^{-1} is given by

$$E(M_1) = e^{2m}\left(\left(1 + \frac{4\sigma^4(1-K_n)^2}{n}\right)e^{2\sigma^2} - \left(1 + \frac{(1-2K_n)^2\,\sigma^4}{n}\right)e^{\sigma^2}\right). \quad \ldots(4.14)$$

Therefore for finite n the estimate M_1 is biased, but asymptotically unbiased.

To the order n^{-1} the percentage of biasedness is given by

$$\frac{(4(1-K_n)^2\,e^{\sigma^2} - (1-2K_n)^2)\,\sigma^4}{n(e^{\sigma^2} - 1)} \times 100. \quad \ldots(4.15)$$

Also from (4.14) when $\rho = 0$ we have

$$E(M_1) = e^{2\alpha_2}\left(\left(1 + \frac{4\sigma_2^4(1-K_n)^2}{n}\right)e^{2\sigma_2^2} - \left(1 + \frac{(1-2K_n)^2\,\sigma_2^4}{n}\right)e^{\sigma_2^2}\right). \quad \ldots(4.16)$$

Equation (4.16) shows that the estimate M_1 shows dependence when there is really independence.

The variance of M_1 to the order n^{-1} is

var $(M_1) =$

$$e^{4m+4\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{16(1-K_n)^2\,\sigma^4}{n}\right)e^{4K_n\sigma^2} - \left(1 + \frac{8\sigma^4(1-K_n)^2}{n}\right)\right)$$

$$+ e^{4m+2\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{4(1-2K_n)^2\,\sigma^4}{n}\right)e^{4K_n\sigma^2} - \left(1 + \frac{2(1-2K_n)^2\,\sigma^2}{n}\right)\right) -$$

(equation continued on p. 821)

$$- 2e^{4m+3\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{(3-4K_n)^2\sigma^4}{n}\right)e^{4K_n\sigma^2} - \left(1 + \frac{(1-2K_n)^2\sigma^4}{n} + \frac{4\sigma^4(1-K_n)^2}{n}\right)\right) \qquad \ldots(4.17)$$

as $n \to \infty$, $\text{var}\,(M_1) \to e^{4m}(e^{4K\sigma^2} - 1)\,(e^{2\sigma^2} - e^{\sigma^2})^2$, but as $K \to 0$, $n \to \infty$,

$\text{var}\,(M_1) \to 0$.

(II) In the second approach we take $g_2(\zeta)$ to be an infinite series given by:

$$g_2(\zeta) = \sum_{s=0}^{\infty}\sum_{r=1}^{\infty} \frac{(2\lambda_n)^s}{s!\,r!}(-1)^{r+1}(\zeta)^{r+s}\,\alpha_{r+s} \qquad \ldots(4.18)$$

where α_{r+s} are independent of ζ and are chosen such that

$$\sum_{s=0}^{\infty}\sum_{r=1}^{\infty} \frac{(2\lambda_n)^s}{s!\,r!}(-1)^{r+1}\alpha_{r+s}E(\zeta)^{r+s}$$

$$= \sum_{s=0}^{\infty}\sum_{r=1}^{\infty} \frac{(2\lambda_n)^s}{s!r!}(-1)^{r+1}(\sigma^2)^{r+s}. \qquad \ldots(4.19)$$

Inserting the value of $E(\zeta)^{r+s}$, we get

$$\alpha_{r+s} = \frac{(n-2)^{r+s}\,\Gamma(\frac{1}{2}(n-2))}{\Gamma((r+s)+\frac{1}{2}(n-2))} \qquad \ldots(4.20)$$

and $g_2(\zeta)$ becomes

$$g_2(\zeta) = \sum_{s=0}^{\infty}\sum_{r=1}^{\infty} \frac{(2\lambda_n)^s}{s!\,r!}(-1)^{r+1}(\zeta)^{r+s}\frac{(n-2)^{r+s}\,\Gamma(\frac{1}{2}(n-2))}{\Gamma((r+s)+\frac{1}{2}(n-2))}. \qquad \ldots(4.21)$$

Hence we get the estimate M_2 for $\text{var}\,(z_2/z_1)$ defined by

$$M_2 = e^{2\hat{\xi}}g_2(\zeta). \qquad \ldots(4.22)$$

The estimate M_2 is an unbiased estimate for $\text{var}\,(z_2/z_1)$, also when $\rho = 0$ we have

$$E(M_2) = e^{2\alpha_2}\,(e^{2\sigma_2^2} - e^{\sigma_2^2}) \qquad \ldots(4.23)$$

i.e. M_2 shows independence when there is really independence.

Expressing $g_2(\zeta)$ in descending powers of n we have

$$g_2(\zeta) = \sum_{s=0}^{\infty} \sum_{r=1}^{\infty} \frac{(4\lambda_n)^s \, 2^r(-1)^{r+1} \zeta^{r+s}}{s! \, r!} \left(1 - (r+s)(r+s-1)\frac{1}{n} + \tfrac{1}{2}(r+s)(r+s-1)(r+s-2)(r+s+7/3)\frac{1}{n^2} + \ldots \right) \quad \ldots(4.24)$$

which can be written as

$$g_2(\zeta) = e^{4\lambda_n \zeta}\left(1 - \frac{(4\lambda_n\zeta)^2}{n} + \frac{(4\lambda_n\zeta)^3 (8 + \sigma\lambda_n\zeta)}{3n^2} + \ldots\right) - e^{2(2\lambda_n - 1)\zeta}\left(1 - \frac{(2(2\lambda_n - 1)\zeta)^2}{n} + \frac{(2(2\lambda_n - 1)\zeta)^3 (8 + 3(2\lambda_n - 1)\zeta)}{3n^2} + \ldots\right). \quad \ldots(4.25)$$

The variance of M_2 to the order n^{-1} is given by

$$\text{var}(M_2) = e^{4m+2\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{2\sigma^4(1 - 2K_n)^2}{n}\right)e^{4K_n\sigma^2} - 1\right) + e^{4m+4\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{8\sigma^4(1 - K_n)^2}{n}\right)e^{4K_n\sigma^2} - 1\right) - 2e^{4m+3\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{4\sigma^4(1 - K_n)(1 - 2K_n)}{n}\right). e^{4K_n\sigma^2} - 1\right). \quad \ldots(4.26)$$

As $n \to \infty$, $\text{var}(M_2) \to e^{4m}(e^{4K\sigma^2} - 1)(e^{2\sigma^2} - e^{\sigma^2})^2$.

But if $K \to 0$, $\text{var}(M_2) \to 0$.

Now it is obvious that the function $g_2(\zeta)$ is very tedious for numerical computation. It is, therefore, suggested that we take $g_2(\zeta)$ expressed to terms of order n^{-1} i.e. we take $g_2(\zeta)$ in the form

$$g_2'(\zeta) = e^{4\zeta(1-K_n)}\left(1 - \frac{4\zeta}{n} - \frac{(4\zeta(1 - K_n))^2}{n}\right) - e^{2\zeta(1-2K_n)}\left(1 - \frac{4\zeta}{n} - \frac{(2(1 - 2K_n)\zeta)^2}{n}\right). \quad \ldots(4.27)$$

Hence we get another M_2' for $\text{var}(z_2/z_1)$ defined by

$$M_2' = \exp(2\hat{\xi} + 2\zeta(1 - 2K_n))\left(\left(1 - \frac{4\zeta}{n} - \frac{(4\zeta(1 - K_n))^2}{n}\right)e^{2\zeta} - \left(1 - \frac{4\zeta}{n} - \frac{(2(1 - 2K_n)\zeta)^2}{n}\right)\right). \quad \ldots(4.28)$$

The expected value of M_2' to the order n^{-1} is given by

$$E(M_2') = e^{2m}(e^{2\sigma^2} - e^{\sigma^2}) \qquad \dots(4.29)$$

i.e. to the order n^{-1} the estimate M_2' is unbiased.

Also when $\rho = 0$ we get

$$E(M_2') = e^{2\alpha_2}(e^{2\sigma_2^2} - e^{\sigma_2^2}) \qquad \dots(4.30)$$

which shows that the estimate M_2' shows independence when there is really independence.

The variance of M_2' to the order n^{-1} is given by

$$\begin{aligned} \text{var}\,(M_2') = {} & e^{4m+2\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{2\sigma^4(1-2K_n)^2}{n}\right)e^{4K_n\sigma^2} - 1\right) \\ & + e^{4m+2\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{8\sigma^4(1-K_n)^2}{n}\right)e^{4K_n\sigma^2} - 1\right) \\ & - 2e^{4m+3\sigma^2}\left(\left(1 + \frac{4\sigma^2}{n} + \frac{4\sigma^4(1-K_n)(1-2K_n)}{n}\right)e^{4K_n\sigma^2} - 1\right) \end{aligned} \qquad \dots(4.31)$$

which is the same as the variance of M_2 i.e. we can use M_2' in large sample without loss of efficiency.

Now if we write $g_2'(\zeta)$ given by (4.27) with form

$$g_2''(\zeta) = \exp(4\zeta(1-K_n))\left(1 - \frac{4\zeta}{n}\right) - \exp(2\zeta(1-2K_n))\left(1 - \frac{4\zeta}{n}\right). \qquad \dots(4.32)$$

Hence we can obtain another estimate

$$\left.\begin{aligned} M_2'' &= e^{2\xi}g_2''(\zeta) \\ M_2'' &= e^{2\hat{\xi}+4\zeta(1-K_n)}\left(1 - \frac{4\zeta}{n}\right) - e^{2\hat{\xi}+2\zeta(1-2K_n)}\left(1 - \frac{4\zeta}{n}\right). \end{aligned}\right\} \qquad \dots(4.33)$$

The expected value of M_2'' to the order n^{-1} is given by

$$E(M_2'') = e^{2m}\left(\left(1 + \frac{4\sigma^4(1-K_n)^2}{n}\right)e^{2\sigma^2} - \left(1 + \frac{(1-2K_n)^2\sigma^4}{n}\right)e^{\sigma^2}\right) \qquad \dots(4.34)$$

i.e. the estimate M_2'' is biased for finite n but asymptotically unbiased. The percentage of biasedness is

$$\frac{(4(1-K_n)^2\, e^{\sigma^2} - (1-2K_n)^2)\,\sigma^4}{n(e^{\sigma^2}-1)} \times 100$$

which is the same as that given by M_1.

The variance of M_2'' to the order n^{-1} is given by

var $(M_2'') =$

$$e^{4m+4\sigma^2}\left(\left(1+\frac{4\sigma^2}{n}+\frac{16(1-K_n)^2\,\sigma^4}{n}\right)e^{4K_n\sigma^2}-\left(1+\frac{8(1-K_n)^2\,\sigma^4}{n}\right)\right)$$
$$+\,e^{4m+2\sigma^2}\left(\left(1+\frac{4\sigma^2}{n}+\frac{4(1-2K_n)^2\,\sigma^4}{n}\right)e^{4K_n\sigma^2}-\left(1+\frac{2(1-2K_n)^2\,\sigma^4}{n}\right)\right)$$
$$-\,2e^{4m+3\sigma^2}\left(\left(1+\frac{4\sigma^2}{n}+\frac{(3-4K_n)^2\,\sigma^4}{n}\right)e^{4K_n\sigma^2}-\left(1+\frac{(1-2K_n)^2\,\sigma^4}{n}\right.\right.$$
$$\left.\left.+\frac{4(1-K_n)^2\,\sigma^4}{n}\right)\right) \qquad \ldots(4.35)$$

which again is the same as that of M_1 given by (4.17).

The efficiency of the estimate M_1 and M_2'' to the order n^{-1} is

$$E_0 = 1 - \frac{\phi_2(n,\sigma^2)}{\phi_1(n,\sigma^2)} \qquad \ldots(4.36)$$

where

$$\phi_2(n,\sigma^2) = \frac{8\sigma^2}{n}\,(e^{4K_n\sigma^2}-1)(e^{\sigma^2}-1)\,((1-K_n)^2\,e^{\sigma^2}-(\tfrac{1}{2}-K_n)^2)$$

and

$$\phi_1(n,\sigma^2) = \phi_2(n,\sigma^2) + (e^{\sigma^2}-1)\,(e^{4K_n\sigma^2}-1+\frac{4\sigma^2}{n}\,e^{4K_n\sigma^2})$$
$$+\,\frac{8\sigma^4}{n}\,e^{4K_n\sigma^2}\,((1-K_n)\,e^{\sigma^2}-(\tfrac{1}{2}-K_n))^2. \qquad \ldots(4.37)$$

It is clear that $\phi_2(n,\sigma^2) \to 0$ as $n \to \infty$.

Hence $E_0 \to 1$ as $n \to \infty$.

We notice that $1 > \dfrac{\phi_2(n,\sigma^2)}{\phi_1(n,\sigma^2)} > 0$.

Then it is easy to show that ϕ_2/ϕ_1 increases as σ^2 increases, i.e. E_0 decreases as σ^2 increases.

It can be shown that the efficiency of the estimate N to the order n^{-1} is less than 1 and tends to $e^{-4K\sigma^2}$ as n tends to ∞. It can also be shown that this efficiency approaches 100% as K tends to zero.

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