Indian J. pure appl. Math., **10**(7) : 826-834, July 1979

A CLASS OF p-VALENT ANALYTIC FUNCTIONS

NIRMAL SINGH SOHI

Department of Mathematics, Punjabi University, Patiala

(*Received* 10 *August* 1978; *after revision* 21 *November* 1978)

Let $S_p(\alpha)$ denote the class of functions $f(z) = z + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ which are regular and p-valent in $|z| < 1$ and satisfy the condition

$$\left|\frac{f'(z)}{pz^{p-1}} - \alpha\right| < \alpha, \quad \alpha > \tfrac{1}{2}.$$

In this paper we obtain distortion theorem, coefficient estimates and radius of convexity for the class $S_p(\alpha)$. It is further proved that the class $S_p(\alpha)$ is closed under convolution.

1. INTRODUCTION

Let $S_p(\alpha)$ denote the class of functions $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ which are regular and p-valent in $|z| < 1$ and which satisfy

$$\left|\frac{f'(z)}{pz^{p-1}} - \alpha\right| < \alpha, \quad \alpha > \tfrac{1}{2}, \quad |z| < 1.$$

Goel (1967) has studied the class $S_1(\alpha)$ and obtained distortion theorems, coefficient bounds and radius of convexity etc. He has further proved that the class $S_1(\alpha)$ is closed under convolution and obtained sharp bounds for arg $f'(z)$, $f(z) \in S_1(\alpha)$ (Goel 1971). Here we shall prove corresponding results for the class $S_p(\alpha)$. By taking $p = 1$, we get the results obtained by Goel (1967, 1971).

2. DISTORTION THEOREM

Theorem 1 — If $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ belongs to the class $S_p(\alpha)$, then

(a) $$p|z|^{p-1}\frac{1-|z|}{1-A|z|} \leqslant |f'(z)| \leqslant p|z|^{p-1}\frac{1+|z|}{1+A|z|}, \quad A = \frac{1}{\alpha} - 1$$

(b) $$\int_0^{|z|} pt^{p-1}\frac{1-t}{1-At}\,dt \leqslant |f(z)| \leqslant \int_0^{|z|} pt^{p-1}\frac{1+t}{1+At}\,dt.$$

The above estimates are sharp.

PROOF : Let

$$\psi(z) = \frac{f'(z)}{\alpha p z^{p-1}} \quad \ldots(1)$$

then

$$\psi(0) = 1/(\alpha - 1). \quad \ldots(2)$$

Again let

$$\phi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\,\psi(z)} \quad \ldots(3)$$

then $\phi(0) = 0$ and $|\phi(z)| < 1$. ...(4)

Therefore by Schwarz's lemma

$$|\phi(z)| \leqslant |z|.$$

From (1), (2) and (3) we get

$$\frac{f'(z)}{pz^{p-1}} = \frac{1 + \phi(z)}{1 + (1/(\alpha - 1))\,\phi(z)}. \quad \ldots(5)$$

From (5) we have in conjunction with (4)

$$\left|\frac{f'(z)}{pz^{p-1}} - \frac{1 - Ar^2}{1 - A^2r^2}\right| \leqslant \frac{(1 - A)\,r}{1 - A^2r^2},\ |z| = r. \quad \ldots(6)$$

The bounds in (a) follow easily from (6).

Integrating along the line segment from 0 to z, we have

$$|f(z)| = \left|\int_0^z f'(s)\,ds\right| \leqslant \int_0^{|z|} |f'(te^{i\theta})|\,d\theta$$

$$\leqslant \int_0^{|z|} pt^{p-1}\frac{1 + t}{1 + At}\,dt.$$

In order to obtain the lower bound for $f(z)$ we integrate along the path L whose image is the line segment $[0, f(z)]$.

$$|f(z)| = \left|\int_L f'(s)\,ds\right| \geqslant \int_L |f'(s)|\,d|s| \geqslant \int_0^{|z|} pt^{p-1}\frac{1 - t}{1 - At}\,dt.$$

This proves the theorem. By taking the function

$$f(z) = \int_0^z pt^{p-1}\frac{1 + t}{1 + At}\,dt$$

one can show that the estimates are sharp.

3. Coefficient Estimates

Theorem 2 — If

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \in S_p(\alpha)$$

then

$$| a_{p+k} | \leqslant \frac{(1 - A)\, p}{p + k}. \qquad \ldots(7)$$

These bounds are sharp for the functions

$$f(z) = \int_0^z p t^{p-1} \frac{1 + t^k}{1 + A t^k}\, dt.$$

PROOF : Since $f(z) \in S_p(\alpha)$, therefore, from (5) of Theorem 1, we have

$$f'(z) = p z^{p-1} \frac{1 + \phi(z)}{1 + A\phi(z)}$$

where $\phi(0) = 0, \ | \phi(z) | < 1$ for $| z | < 1$.

Therefore,

$$\phi(z) \left[p z^{p-1} - A f'(z) \right] = f'(z) - p z^{p-1}$$

or

$$\phi(z) \left[(1 - A)\, p z^{p-1} - A \sum_{k=1}^{\infty} (p + k)\, a_{p+k} z^{p+k-1} \right] = \sum_{k=1}^{n+1} (p + k)\, a_{p+k} z^{p+k-1}$$

or

$$\phi(z) \left[(1 - A)\, p z^{p-1} - A \sum_{k=1}^{n} (p + k)\, a_{p+k} z^{p+k-1} \right] = \sum_{k=1}^{n+1} (p + k)\, a_{p+k} z^{p+k-1} + \sum_{k=n+1}^{\infty} c_k z^{p+k}$$

c_k being some complex numbers. Since $| \phi(z) | < 1$, we have by means of Parsavel's identity

$$\sum_{k=1}^{n+1} (p + k)^2 \, | a_{p+k} |^2 r^{2(p+k-1)} + \sum_{k=n+1}^{\infty} | c_k |^2 r^{2(p+k)} \leqslant (1 - A)^2 p^2 r^{2(p-1)} + A \sum_{k=1}^{n} (p + k)^2 \, | a_{p+k} |^2 r^{2(p+k-1)}.$$

Hence further

$$\sum_{k=1}^{n+1} (p+k)^2 \mid a_{p+k} \mid^2 \leqslant (1-A)^2 p^2 - (1-A^2) \sum_{k=1}^{n} (p+k)^2 \mid a_{p+k} \mid^2.$$

or

$$(p+n+1)^2 \mid a_{p+n+1} \mid^2 \leqslant (1-A)^2 p^2$$

or

$$\mid a_{p+n+1} \mid \leqslant \frac{(1-A)\,p}{p+n+1}$$

or

$$\mid a_{p+k} \mid \leqslant \frac{(1-A)\,p}{p+k}.$$

Consider the function

$$f(z) = \int_0^z p t^{p-1} \frac{1+t^k}{1+At^k}, \quad \mid z \mid < 1.$$

Then

$$\left| \frac{1}{\alpha} \frac{f'(z)}{pz^{p-1}} - 1 \right| < 1, \quad \mid z \mid < 1$$

and the function $f(z)$ has the expansion

$$f(z) = z^p + \frac{(1-A)\,p}{p+k} z^{p+k} + \dots \quad \text{for} \quad \mid z \mid < 1$$

showing that the estimate is sharp.

4. Convex Set of Functions

Theorem 3 — If $f(z)$ and $g(z)$ belong to the class $S_p(\alpha)$ i.e.

$$\left| \frac{f'(z)}{\alpha p z^{p-1}} - 1 \right| < 1, \left| \frac{g'(z)}{\alpha p z^{p-1}} - 1 \right| < 1$$

then $\quad \lambda f(z) + (1-\lambda)\, g(z), \quad (0 \leqslant \lambda \leqslant 1)$

belongs to the class $S_p(\alpha)$.

Proof : $$\left| \frac{\lambda f'(z) + (1-\lambda)\, g'(z)}{\alpha p z^{p-1}} - 1 \right| \leqslant \lambda \left| \frac{f'(z)}{\alpha p z^{p-1}} - 1 \right| + (1-\lambda) \left| \frac{g'(z)}{\alpha p z^{p-1}} - 1 \right|$$
$$\leqslant \lambda + (1-\lambda) = 1 \quad \text{for} \quad \mid z \mid < 1.$$

This proves that $\lambda f(z) + (1-\lambda)\, g(z) \in S_p(\alpha)$.

5. Radius of Convexity

Theorem 4 — Each function in $S_p(\alpha)$ is convex in $|z| < R$, where

$$R = \frac{[p(1+A)+1-A] - \sqrt{[p(1+A)+1-A]^2 - 4Ap^2}}{2pA},$$

and the result is best possible.

Proof: From (5) of Theorem 1, we have

$$f'(z) = pz^{p-1}\,\frac{1+\phi(z)}{1+A\phi(z)}.$$

Taking logarithmic derivative, we get after some simplification

$$1 + \frac{zf''(z)}{f'(z)} = \frac{(1-A)\,z\phi'(z)}{(1+\phi(z))\,(1+A\phi(z))}.$$

Putting $\phi(z) = zh(z)$, so that $h(z)$ is analytic and $|h(z)| \leqslant 1$ for $|z| < 1$, we have

$$\text{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geqslant p - \frac{(1-A)\,|z|\,(|h(z)| + |z|)\,(1-|zh(z)|)}{(1-|z|^2)\,|(1+zh(z))\,(1+Azh(z))|}$$

where we have used the estimate

$$|h'(z)| \leqslant \frac{1-|h(z)|^2}{1-|z|^2} \quad \text{(Nehari 1952, p. 168).}$$

Now, for $\alpha \geqslant 1$

$$\begin{aligned} |(1+zh(z))\,(1+Azh(z))| &= |1 + (1+A)\,zh(z) + Az^2h^2(z)| \\ &\geqslant 1 - (1+A)\,|zh(z)| + A\,|zh(z)|^2 \\ &= (1-|zh(z)|)\,(1-A\,|zh(z)|). \end{aligned}$$

On the other hand, for $\frac{1}{2} < \alpha \leqslant 1$,

$$|(1+zh(z))\,(1+Azh(z))| \geqslant (1-|zh(z)|)\,(1-A\,|zh(z)|).$$

Hence in either case, we have

$$\text{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geqslant p - \frac{(1-A)\,|z|}{(1-|z|)\,(1-A\,|z|)}.$$

Thus $f(z)$ is convex if

$$|z| < \frac{[p(1+A)+1-A] - [(p(1+A)+1-A)^2 - 4Ap^2]^{1/2}}{2pA}.$$

To see that the result is sharp, take

$$f'(z) = pz^{p-1}\,\frac{1+z}{1+Az}.$$

Then

$$1 + \frac{zf''(z)}{f'(z)} = p + \frac{(1-A)z}{(1+Az)(1+z)} = 0$$

when $|z| = -R$, and this proves the theorem.

6. Argument of $f'(z)/pz^{p-1}$

Theorem 5 — For $f(z) \in S_p(\alpha)$, we have

$$\left| \arg \frac{f'(z)}{pz^{p-1}} \right| \leqslant \sin^{-1} \frac{(1-A)|z|}{1-A|z|^2}$$

and the bound is sharp.

PROOF: From (6) Theorem 1, we have

$$\left| \frac{f'(z)}{pz^{p-1}} - \frac{1-A|z|^2}{1-A^2|z|^2} \right| \leqslant \frac{(1-A)|z|}{1-A^2|z|^2}.$$

Therefore,

$$\left| \arg \frac{f'(z)}{pz^{p-1}} \right| \leqslant \sin^{-1} \frac{(1-A)|z|}{1-A|z|^2}.$$

To see that the result is sharp, let

$$\frac{f'(z)}{pz^{p-1}} = \frac{1+\epsilon z}{1+\epsilon Az}, \quad |\epsilon| = 1.$$

Putting $\epsilon = \frac{r}{2}\left[-\frac{(1+A)r}{1+Ar^2} \pm \frac{i\sqrt{(1-r^2)(1-A^2r^2)}}{1+Ar^2} \right], \quad r = |z|$

we have

$$\arg \frac{f'(z)}{pz^{p-1}} = \sin^{-1} \frac{(1-A)r}{1-Ar^2}.$$

7. Convolution

Theorem 6 — If

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$

and

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$$

belong to $S_p(\alpha)$, then

$$F(z) = z^p + \tfrac{1}{2} \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) a_{p+k} b_{p+k} z^{p+k}$$

is also a member of $S_p(\alpha)$.

PROOF: Since $f(z)$ and $g(z)$ belong to $S_p(\alpha)$, therefore

$$\left| \frac{f'(z)}{pz^{p-1}} - \alpha \right| < \alpha, \quad \text{for } |z| < 1$$

$$\left| \frac{g'(z)}{pz^{p-1}} - \alpha \right| < \alpha, \quad \text{for } |z| < 1.$$

It is well-known that if $h(z) = \sum_{n=0}^{\infty} C_n z^n$ is regular for $|z| < 1$ and $|h(z)| \leqslant M$ then

$$\sum_{n=0}^{\infty} |C_n|^2 \leqslant M^2.$$

Applying this estimate to the functions $\frac{f'(z)}{pz^{p-1}} - \alpha$ and $\frac{g'(z)}{pz^{p-1}} - \alpha$, we get

$$(1-\alpha)^2 + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 |a_{p+k}|^2 \leqslant \alpha^2$$

and

$$(1-\alpha)^2 + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 |b_{p+k}|^2 \leqslant \alpha^2$$

or

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^2 |a_{p+k}|^2 \leqslant 2\alpha - 1$$

and

$$\sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right) |b_{p+k}|^2 \leqslant 2\alpha - 1, \quad \alpha > \tfrac{1}{2}.$$

Now

$$\left|\frac{F'(z)}{pz^{p-1}}-\alpha\right|^2=\left|(1-\alpha)+\tfrac{1}{2}\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2 a_{p+k}b_{p+k}z^k\right|^2$$

$$\leqslant(1-\alpha)^2+(1-\alpha)\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|a_{p+k}|\,|b_{p+k}|\,r^k$$

$$+\tfrac{1}{4}\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|a_{p+k}|\,|b_{p+k}|\,r^k\right)^2,\quad |z|=r$$

$$\leqslant(1-\alpha)^2+(1-\alpha)\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|a_{p+k}|\,|b_{p+k}|$$

$$+\tfrac{1}{4}\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|a_{p+k}|\,|b_{p+k}|\right)^2$$

$$\leqslant(1-\alpha)^2+(1-\alpha)\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|a_{p+k}|\right)^{1/2}$$

$$\times\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|b_{p+k}|^2\right)^{1/2}$$

$$+\tfrac{1}{4}\left(\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|a_{p+k}|^2\right.$$

$$\left.\times\sum_{k=1}^{\infty}\left(\frac{p+k}{p}\right)^2|b_{p+k}|^2\right)$$

$$\leqslant(1-\alpha)^2+(1-\alpha)(2\alpha-1)+\tfrac{1}{4}(2\alpha-1)^2.$$

Consequently $\left|\dfrac{F'(z)}{pz^{p-1}}-\alpha\right|^2<\alpha^2$

if $(1-\alpha)^2+(1-\alpha)(2\alpha-1)+\frac{1}{4}(2\alpha-1)^2<\alpha^2$
that is, if

$$\tfrac{1}{4}(1-2\alpha)(2\alpha+1)<0$$

which is satisfied for $\alpha>\frac{1}{2}$.

Hence $F(z)\in S_p(\alpha)$.

ACKNOWLEDGEMENT

The author's thanks are due to Dr R. M. Goel for his kind encouragement and guidance. He also wishes to thank the refree for his valuable suggestions.

REFERENCES

Goel, R. M. (1967). A class of univalent functions whose derivatives have positive real part in the unit disc. *Nieuw Anchief voor Wiskunde* (3), XV, 55-63.

——————— (1971). A class of analytic functions whose derivatives have positive real in the unit disc. *Indian J. Math.*, **13**, No. 3, 141-45.

Nehari, Z. (1952). Conformal Mapping. McGraw-Hill Book Co., Inc., New York.