Indian J. pure appl. Math., **10**(7) : 835-841, July 1979

CONFORMAL TRANSFORMATIONS ASSOCIATED WITH SETS OF n FUNDAMENTAL FORMS OF RIEMANNIAN HYPERSURFACE

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(*Received* 29 *August* 1978; *after revision* 25 *November* 1978)

In the present paper the relations between the pth fundamental tensors of two spaces which are hypersurfaces of conformally related Riemannian spaces have been obtained. The correspondence between the pth curvatures and the pth associated mean curvatures of these hypersurfaces have also been established.

1. FUNDAMENTAL FORMULAE

Let a $n-1$ dimensional hypersurface V_{n-1} given by the equations

$$x^i = f^i(u^\alpha) \quad (i = 1, \ldots, n;\ \alpha = 1, \ldots, n-1) \qquad \ldots(1.1)$$

be immersed in a Riemannian space V_n. Let us suppose that the functions (1.1) are at least of class C^3 in u^α and the projection factors $B^j_\alpha = \dfrac{\partial f^j}{\partial u^\alpha}$ are such that their matrix has maximal rank $n-1$. We shall also write $B^j_{\alpha\beta} = \dfrac{\partial^2 f^j}{\partial u^\alpha \partial u^\beta} = B^j_{\beta\alpha}$. If $g_{hj}(x^k)$ denotes the metric tensor of V_n, the induced metric tensor of V_{n-1} is defined as

$$g_{\alpha\beta} = g_{hj} B^h_\alpha\ B^j_\beta . \qquad \ldots(1.2)$$

The inverse of (1.2) is denoted by $g^{\alpha\beta}$, by means of which we define the quantities $B^\alpha_j = g^{\alpha\epsilon} B^h_\epsilon\ g_{hj}$ for which $B^\alpha_j\ B^j_\beta = \delta^\alpha_\beta$. The unit normal vector N^j of V_{n-1} is determined uniquely by the relations

$$\text{(a)} \quad g_{hj} B^h_\beta\ N^j = 0, \quad \text{(b)} \quad g_{hj} N^h N^j = 1. \qquad \ldots(1.3)$$

We have the following identity

$$B^i_\alpha\ B^\alpha_j = \delta^i_j - N^i N_j. \quad (N_j = g_{ij} N^i). \qquad \ldots(1.4)$$

The symmetric induced connection coefficients (i.e. the Christoffel symbols) of V_{n-1} are defined by

$$\Gamma^\alpha_{\beta\gamma} = B^\alpha_j\,(B^j_{\beta\gamma} + \Gamma^j_{hk}\, B^h_\beta\, B^k_\gamma) \qquad \ldots(1.5)$$

in which Γ^j_{hk} denote the Christoffel symbols of V_n. By means of these symbols the mixed covariant derivative of the projection factors B^j_β has been defined as

$$I^j_{\beta\gamma} = B^j_{\beta\gamma} - B^j_\alpha \Gamma^\alpha_{\beta\gamma} + \Gamma^j_{hk} B^h_\beta B^k_\gamma . \qquad ...(1.6)$$

From (1.5) it follows that $I^j_{\beta\gamma}$ is normal to V_{n-1}. Therefore, we may write

$$I^j_{\beta\gamma} = \Omega_{\beta\gamma} N^j \qquad ...(1.7)$$

in which the factors of proportionality $\Omega_{\alpha\beta}$ is symmetric in α and β by virtue of (1.6). These quantities are called the coefficients of the second fundamental form of V_{n-1}. In general the coefficients of the pth fundamental form of V_{n-1} are defined by (Rund 1971)

$$C_{(1)\alpha\beta} = g_{\alpha\beta}$$

$$C_{(2)\alpha\beta} = \Omega_{\alpha\beta}$$

$$C_{(p)\alpha\beta} = C_{(p-1)\alpha\varepsilon}\Omega^\varepsilon_\beta \quad (p = 2, 3, ..., n) \qquad ...(1.8)$$

where $\quad \Omega^\varepsilon_\beta = g^{\alpha\varepsilon}\Omega_{\alpha\beta}$.

2. The Conformal Correspondence

Let V^*_n be a Riemannian space which is conformal to V_n. If g^*_{hj} denote the metric tensors of V^*_n, then we have

$$g^*_{hj} = e^{2\sigma} g_{hj}, \; g^{*hj} = e^{-2\sigma} g^{hj}, \qquad ...(2.1)$$

where $\sigma = \sigma(x)$ is a scalar function. If σ is constant then the correspondence of V_n and V^*_n is said to be homothetic.

Let V^*_{n-1} be a hypersurface of V^*_n given by the equation

$$x^i = g^i(u^\alpha). \qquad ...(2.2)$$

Let us suppose that the projection factors $B^{*j}_\alpha = \dfrac{\partial g^j}{\partial u^\alpha}$ are such that their matrix has the maximal rank $n - 1$. The induced metric tensor of V^*_{n-1} is defined as

$$g^*_{\alpha\beta} = g^*_{hj} B^{*h}_\alpha B^{*j}_\beta . \qquad ...(2.3)$$

If the functions (1.1) and (2.2) are such that $f^i - g^i$ is a constant function i.e.

$$B^{*i}_{\alpha} = B^{i}_{\alpha}, \qquad \ldots(2.4)$$

then from (1.2), (2.1), (2.3) and (2.4) it follows that

$$g^{*}_{\alpha\beta} = e^{2\sigma} g_{\alpha\beta}. \qquad \ldots(2.5)$$

Hence we have the following proposition :

Proposition 1 — Let V_n and V^*_n are two conformally related Riemannian spaces with the conformal coefficient $e^{2\sigma}$. If V_{n-1} and V^*_{n-1} are hypersurfaces of V_n and V^*_n respectively, represented by eqns. (1.1) and (2.2), then a sufficient condition that these hypersurfaces are conformally related with the same conformal coefficient $e^{2\sigma}$ is that $f^i - g^i$ is a constant function.

In the following we assume that the functions (1.1) and (2.2) differ only by a constant function i.e. the equality (2.4) is satisfied. We shall denote the quantities of V^*_n and V^*_{n-1} by star letters.

From (2.1) and (2.5) we have the following :

$$\Gamma^{*j}_{hk} = \Gamma^{j}_{hk} + \delta^{j}_{h}\, \sigma_k + \delta^{j}_{k}\, \sigma_h - g_{hk}\sigma^j \qquad \ldots(2.6)$$

$$\Gamma^{*\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \delta^{\alpha}_{\beta}\, \sigma_\gamma + \delta^{\alpha}_{\gamma}\, \sigma_\beta - g_{\beta\gamma}\sigma^\alpha \qquad \ldots(2.7)$$

where

$$\sigma_k = \frac{\partial\sigma}{\partial x^k},\ \sigma_\gamma = \frac{\partial\sigma}{\partial u^\gamma} = B^{k}_{\gamma}\, \sigma_k,\ \sigma^j = g^{jh}\sigma_h,\ \sigma^\alpha = g^{\alpha\beta}\sigma_\beta.$$

Let N^{*j} denote the unit normal vector of V^*_{n-1}. Then

$$g^{*}_{hj}\, B^{h}_{\beta}\, N^{*j} = 0,\ g^{*}_{hj}\, N^{*h}N^{*j} = 1. \qquad \ldots(2.8)$$

From (2.1), (1.3) and (2.8) it follows that the vector N^{*j} is also normal to V_{n-1} whose magnitude in V_n is $e^{-\sigma}$. Thus we have

$$N^{*j} = e^{-\sigma}N^j. \qquad \ldots(2.9)$$

Taking tensor derivatives of $B^{i}_{\alpha} = B^{*i}_{\alpha}$ in their respective spaces and using the relations (1.4), (1.6), (2.6) and (2.7) we get

$$I^{*j}_{\beta\gamma} = I^{j}_{\beta\gamma} - \mu g_{\beta\gamma} N^j \qquad \ldots(2.10)$$

where

$$\mu = N^i \sigma_i.$$

From (1.7), (2.9) and (2.10) it follows that

$$\Omega^*_{\beta\gamma} = e^{\sigma}(\Omega_{\beta\gamma} - \mu g_{\beta\gamma}). \quad ...(2.11)$$

If $\mu = 0$ then either σ is constant or σ_i is tangential to the hypersurface V_{n-1}. Hence we have the following propositions.

Proposition 2 — The normal curvature vector $I^j_{\beta\gamma}$ is invariant under the conformal transformation σ if and only if σ_i is tangential to the hypersurface V_{n-1} or the correspondence of V_n and $\overset{*}{V}_n$ is homothetic or V_n is umbilical.

Proposition 3 — The second fundamental tensors of $\overset{*}{V}_{n-1}$ and V_{n-1} are proportional if and only if σ_i is tangential to the hypersurface V_{n-1} or the correspondence of V_n and $\overset{*}{V}_n$ is homothetic.

Let t^{α} represent a principal direction of V_{n-1} at P so that

$$(\Omega_{\beta\gamma} - \lambda g_{\beta\gamma})\, t^{\beta} = 0, \quad ...(2.12)$$

in which the real eigenvalue λ is the normal curvature of V_{n-1} for this particular direction. It then follows from (2.5), (2.11) and (2.12) that

$$(\Omega^*_{\beta\gamma} - (\lambda - \mu)\, e^{-\sigma} g^*_{\beta\gamma})\, t^{\beta} = 0. \quad ...(2.13)$$

Hence we have the following theorem :

Theorem 1 — Let V_{n-1} be a hypersurface of a Riemannian space V_n and $\overset{*}{V}_n$ is conformal to V_n. If t^{α} is a principal direction of V_{n-1} at P with the normal curvature λ then t^{α} is also a principal direction of the hypersurface $\overset{*}{V}_{n-1}$ of $\overset{*}{V}_n$ at the corresponding point P^* with the normal curvature $(\lambda - \mu)\, e^{-\sigma}$.

From (2.11) and (2 5) it is clear that

$$\Omega^{*\epsilon}_{\beta} = e^{-\sigma}(\Omega^{\epsilon}_{\beta} - \mu \delta^{\epsilon}_{\beta}). \quad ...(2.14)$$

We establish the following theorem :

Theorem 2 — The coefficients of pth fundamental form of $\overset{*}{V}_{n-1}$ and V_{n-1} are related by

$$C^*_{(p)\alpha\beta} = \sum_{m=0}^{p-1} e^{(3-p)\sigma}(-1)^m \binom{p-1}{m} \mu^m C_{(p-m)\alpha\beta} (2 \leqslant p \leqslant n),$$

$$C^*_{(1)\alpha\beta} = e^{2\sigma} C_{(1)\alpha\beta}. \quad \ldots(2.15)$$

PROOF: The validity of this assertion is established by induction. From (1.8), (2.5) and (2.11) it is immediately evident that (2.15) holds for $p = 2$. For a given fixed value of the integer s, with $2 \leqslant s < n$ we have

$$C^*_{(s+1)\alpha\beta} = C^*_{(s)\alpha\varepsilon} \Omega^{*\varepsilon}_{\beta} .$$

Now let us suppose that (2.15) is valid for $p = 2, 3, \ldots s$ so that we have from (2.14) and (2.15)

$$C^*_{(s+1)\alpha\beta} = \sum_{m=2}^{s-1} e^{(3-s)\sigma}(-1)^m \binom{s-1}{m} \mu^m C_{(s-m)\alpha\varepsilon} e^{-\sigma}(\Omega^{\varepsilon}_{\beta} - \mu\delta^{\varepsilon}_{\beta})$$

$$= \sum_{m=0}^{s} e^{(3-s-1)\sigma}(-1)^m \binom{s}{m} \mu^m C_{(s+1-m)\alpha\beta}.$$

This shows that (2.15) is valid for $p = s + 1$. Hence the theorem is established.

We discuss the following particular cases :

Case I — If σ_i is tangential to the hypersurface $V_{n-1}(\mu = 0)$, then we have from (2.15)

$$C^*_{(p)\alpha\beta} = e^{(3-p)\sigma} C_{(p)\alpha\beta} \quad (1 \leqslant p \leqslant n)$$

Hence we have the following :

Proposition 4 — If σ_i is tangential to the hypersurface V_{n-1}, then the pth fundamental tensors $(1 \leqslant p \leqslant n)$ of V_{n-1} is proportional to the pth fundamental tensors of the conformal hypersurface V^*_{n-1}.

Case II — If the space V^*_n is homothetic to V_n then $\sigma_i = 0$ and hence $\mu = 0$. Thus

Proposition 5 — If a Riemannian space V^*_n is homothetic to the space V_n, then the pth fundamental tensor $(1 \leqslant p \leqslant n)$ of V^*_{n-1} is proportional to the pth fundamental tensor of V_{n-1}.

Case III — If V_{n-1} is an umblical hypersurface of V_n with mean curvature μ, then from (2.11) and (1.8) it follows that $C^*_{(2)\alpha\beta} = 0$ and hence $C^*_{(p)\alpha\beta} = 0$. Thus

Proposition 6 — If V_{n-1} is an umbical hypersurface of V_n with mean curvature μ, then the pth fundamental tensors $(2 \leqslant p \leqslant n)$ of the conformal hypersurface V^*_{n-1} vanishes.

The eigenvalues of Ω^α_β are given by the polynomial

$$\det(\Omega^\alpha_\beta - \lambda\delta^\alpha_\beta) = 0. \qquad ...(2.16)$$

From (2.14) it follows that

$$\det[\Omega^{*\alpha}_\beta - (\lambda - \mu)\, e^{-\sigma}\delta^\alpha_\beta] = 0. \qquad ...(2.17)$$

Thus if $\lambda_1, \lambda_2, ..., \lambda_{n-1}$ are the eigenvalues of Ω^α_β, then $\theta_1, \theta_2, ..., \theta_{n-1}$ will be the eigenvalues of $\Omega^{*\alpha}_\beta$ where $\theta_p = (\lambda_p - \mu)\, e^{-\sigma}(1 \leqslant p \leqslant n-1)$.

Let H_p represent the pth curvature of V_{n-1} i.e.

$$H_1 = \lambda_1 + \lambda_2 + ... + \lambda_{n-1},$$
$$H_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + ... + \lambda_{n-2}\lambda_{n-1},$$
$$... \quad ... \quad ... \quad ...$$
$$H_{n-1} = \lambda_1\lambda_2 ... \lambda_{n-1}.$$

If H^*_p is the pth curvature of V^*_{n-1}, then as a direct calculation we have the following.

Theorem 3 — The pth curvature of V^*_{n-1} and V_{n-1} are related by

$$H^*_p = \sum_{m=0}^{p} (-1)^m e^{-p\sigma} \binom{n+m-p-1}{m} \mu^m H_{p-m} (1 \leqslant p \leqslant n-1) \qquad ...(2.18)$$

where $H_0 = 1$.

If σ_i is tangential to the hypersurface V_{n-1}, then we have from (2.18) $H^*_p = e^{-p\sigma} H_p$. Hence we have the following :

Proposition 7 — If σ_i is tangential to the hypersuface V_{n-1}, then the pth curvature of V^*_{n-1} is proportional to the pth curvature of V_{n-1}.

Theorem 4 — The pth associated mean curvature of V^*_{n-1} and V_{n-1} are related by

$$M^*_{(p)} = \sum_{m=0}^{p} e^{-p\sigma}(-1)^m \binom{p}{m} \mu^m M_{(p-m+1)} (1 \leqslant p \leqslant n-1). \qquad ...(2.19)$$

PROOF : The pth associated mean curvature of V_{n-1} is defined by (Rund 1971)

$$M_{(p)} = g^{\alpha\beta} C_{(p+1)\alpha\beta} \quad (p = 1, 2, ..., n-1). \qquad ...(2.20)$$

From (2.15) and (2.5) we get the relation (2.19).

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