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ON COEFFICIENT BOUNDS OF p-VALENT λ-SPIRAL FUNCTIONS OF ORDER α

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Let $S^{\lambda}(p, \alpha)$ be the family of functions f of p-valent λ-spiral functions of order α, f is holomorphic in $E = \{z : |z| < 1\}$ and of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k.$$

We obtain here the bounds of a_n and maximization of $|a_{p+2} - \mu a_{p+1}^2|$ over the class $S^{\lambda}(p, \alpha)$ for real and complex values of μ.

1. INTRODUCTION

Let p be a fixed integer greater than zero; $E = \{z : |z| < 1\}$ is the open unit disc in the complex plane. A function f of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad \ldots(1.1)$$

that is regular in E and satisfies the inequality

$$\operatorname{Re}\left(e^{i\lambda} \frac{zf'(z)}{f(g)}\right) > \alpha \cos \lambda \qquad \ldots(1.2)$$

for $z \in E$, $|\lambda| < (\pi/2)$ and $0 \leqslant \alpha < p$, is called p-valent λ-spiral function of order α. We denote the class of all p-valent λ-spiral functions of order α by $S^{\lambda}(p, \alpha)$.

Clearly for $\lambda = 0$, we have the class $S^*(p, \alpha)$ of p-valent starlike functions of order α considered by Goodman (1950); for $\alpha = 0$ we have p-valent λ-spiral function and further for $p = 1$ we get univalent λ-spiral functions investigated by Libera (1967).

Let B be the class of functions w which are regular in E and satisfy the conditions $\omega(0) = 0$ and for which $|\omega(z)| < 1$ for $z \in E$; in what follows, ω will always have this meaning.

Let $\mathcal{P}(p, \alpha)$, ($0 \leqslant \alpha < p$ with p a positive integer) denote the class of functions with positive real part of order α that have the form

$$P(z) = p + \sum_{k=1}^{\infty} c_k z^k \quad \ldots(1.3)$$

which are regular in E and satisfy the conditions $P(0) = p$ and $\text{Re } P(z) > \alpha (0 \leqslant \alpha < p)$ in E.

The purpose of this paper is to obtain coefficient bounds of p-valent λ-spiral functions of order α. Theorem 3.1 is an extension of Theorem 1 of Libera (1967) for p-valent case; while Theorem 3.2 is concerned with maximization of

$$| a_{p+2} - \mu a_{p+1}^2 |$$

over the class $S^\lambda(p, \alpha)$ for a given real as well as a complex number μ.

We state below some lemmas that are needed in our investigation.

2. Some Lemmas

The following lemma is to be found in Nehari (1952, p. 172).

Lemma 2.1 — If $\omega(z) \in B$, then $| \omega(z) | \leqslant | z |$ and that if

$$\omega(z) = \sum_{k=1}^{\infty} b_k z^k$$

then $| b_1 | \leqslant 1$

and

$$| b_2 | \leqslant 1 - | b_1 |^2. \quad \ldots(2.1)$$

The following lemma is due to Keogh and Merkes (1969), the proof of which may be given by using Lemma 2.1.

Lemma 2.2 — Let $\omega(z) = \sum_{k=1}^{\infty} b_k z^k$ be analytic with $| \omega(z) | < 1$ in E. If ν is any complex number then

$$| b_2 - \nu b_1^2 | \leqslant \max (1, | \nu |). \quad \ldots(2.2)$$

Equality may be attained with functions $\omega(z) = z^2$ and $\omega(z) = z$.

We also need the following lemma.

Lemma 2.3 — The function $P \in \mathcal{P}(p, \alpha)$ if and only if

$$P(z) = \frac{p - (p - 2\alpha)\, \omega(z)}{1 + \alpha(z)} \quad \ldots(2.3)$$

where $\omega(z) = z\phi(z)$ and ϕ is regular function bounded by 1 in E.

Next we state the following lemma.

Lemma 2.4 — If $f \in S^{\lambda}(p, \alpha)$ then

$$e^{i\lambda} \sec \lambda \frac{zf'(z)}{f(z)} - ip \tan \lambda = \frac{p - (p - 2\alpha)\, \omega(z)}{1 + \omega(z)}. \qquad \text{...(2.4)}$$

PROOF : If $f \in S^{\lambda}(p, \alpha)$, then the introduction of appropriate normalizing factors enables us to write

$$\left. \frac{e^{i\lambda}(zf'(z)/f(z)) - ip \sin \lambda}{\cos \lambda} \right|_{z=0} = p.$$

This form helps to represent the members of $S^{\lambda}(p, \alpha)$ in terms of functions of class $P(p, \alpha)$. Thus f is p-valent λ-spiral function of orders α if and only if there exists a function $P \in \mathcal{P}(p, \alpha)$ such that

$$e^{i\lambda} \frac{zf'(z)}{f(z)} = \cos \lambda\, P(z) + ip \sin \lambda.$$

Use of Lemma 2.3 leads to the result of this lemma.

We also need the following.

Lemma 2.5 — If integers p and m are greater than zero; $0 \leqslant \alpha < p$ and $|\lambda| < \pi/2$, then

$$\prod_{j=0}^{m-1} \frac{|2(p-\alpha)\cos\lambda\, e^{-i\lambda} + j|^2}{(j+1)^2} = \frac{4(p-\alpha)\cos^2\lambda}{m^2}$$
$$\times \left\{ (p-\alpha) + \sum_{k=1}^{m-1} (p+k-\alpha) \prod_{j=0}^{k-1} \frac{|2(p-\alpha)\cos\lambda\, e^{-i\lambda} + j|^2}{(j+1)^2} \right\}. \qquad \text{...(2.5)}$$

PROOF : We prove the lemma by induction on m. For $m = 1$, the lemma is obvious.

Next suppose that the result is true for $m = q - 1$. We have

$$\frac{4(p-\alpha)\cos^2\lambda}{q^2}$$
$$\times \left\{ (p-\alpha) + \sum_{k=1}^{q-1} (p+k-\alpha) \prod_{j=0}^{k-1} \frac{|2(p-\alpha)\cos\lambda\, e^{-i\lambda} + j|^2}{(j+1)^2} \right\} =$$

(*equation continued on p.* 845)

$$= \frac{4(p-\alpha)\cos^2\lambda}{q^2}$$

$$\times \Bigg\{(p-\alpha) + \sum_{k=1}^{q-2}(p+k-\alpha)\prod_{j=0}^{k-1}\frac{|2(p-\alpha)\cos\lambda\, e^{-i\lambda}+j|^2}{(j+1)^2}$$

$$+ (p+q-1-\alpha)\prod_{j=0}^{q-2}\frac{|2(p-\alpha)\cos\lambda\, e^{-i\lambda}+j|^2}{(j+1)^2}\Bigg\}$$

$$= \prod_{j=0}^{q-2}\left|\frac{2(p-\alpha)\cos\lambda\, e^{-i\lambda}+j}{j+1}\right|^2$$

$$\times \left\{\frac{(q-1)^2 + 4(p-\alpha)(p+q-1-\alpha)\cos^2\lambda}{q^2}\right\}$$

$$= \prod_{j=0}^{q-1}\left|\frac{2(p-\alpha)\cos\lambda\, e^{-i\lambda}+j}{j+1}\right|^2.$$

Showing that the result is valid for $m = q$; and we are done.

3. Main Theorems

Theorem 3.1 — If $f \in S^{\lambda}(p, \alpha)$, then

$$|a_n| \leqslant \prod_{k=0}^{n-(p+1)}\frac{|2(p-\alpha)\cos\lambda\, e^{-i\lambda}+k|}{k+1} \qquad \ldots(3.1)$$

for $n \geqslant p+1$ and these bounds are sharp for all admissible λ and α and for each n.

Proof: As $f \in S^{\lambda}(p, \alpha)$, from Lemma 2.4, we have

$$e^{i\lambda}\sec\lambda\,\frac{zf'(z)}{f(z)} - ip\tan\lambda = \frac{p-(p-2\alpha)\,\omega(z)}{1+\omega(z)}.$$

This may be written as

$$\{e^{i\lambda}\sec\lambda\,(zf'(z)) + (p-2\alpha-ip\tan\lambda)\,f(z)\}\,\omega(z)$$
$$= (p+ip\tan\lambda)\,f(z) - e^{i\lambda}\sec\lambda\,(zf'(z)).$$

Hence

$$\Bigg[e^{i\lambda}\sec\lambda\left\{pz^p + \sum_{k=1}^{\infty}(p+k)\,a_{p+k}z^{p+k}\right\}$$
$$+ (p-2\alpha-ip\tan\lambda)\left\{z^p + \sum_{k=1}^{\infty}a_{p+k}z^{p+k}\right\}\Bigg]\,\omega(z) =$$

(*equation continued on p.* 846)

$$= (p + ip \tan \lambda) \left\{ z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \right\}$$

$$- e^{i\lambda} \sec \lambda \left\{ pz^p + \sum_{k=1}^{\infty} (p + k) a_{p+k} z^{p+k} \right\}$$

or

$$\left[pe^{i\lambda} \sec \lambda + (p - 2\alpha - ip \tan \lambda) + \sum_{k=1}^{\infty} \{(p + k) e^{i\lambda} \sec \lambda + (p - 2\alpha - ip \tan \lambda)\} a_{p+k} z^k \right] \omega(z)$$

$$= p + ip \tan \lambda - pe^{i\lambda} \sec \lambda + \sum_{k=1}^{\infty} \{p + ip \tan \lambda - (p + k) e^{i\lambda} \sec \lambda\} a_{p+k} z^k$$

which may be written as

$$\sum_{k=0}^{\infty} [\{(p + k) e^{i\lambda} \sec \lambda + (p - 2\alpha - ip \tan \lambda)\} a_{p+k} z^k] \omega(z)$$

$$= \sum_{k=0}^{\infty} [p + ip \tan \lambda - (p + k) e^{i\lambda} \sec \lambda] a_{p+k} z^k \qquad \ldots(3.2)$$

where $\quad a_p = 1 \quad$ and $\quad \omega(z) = \sum_{k=0}^{\infty} b_{k+1} z^{k+1}$.

Equating coefficients of z^m on both sides of (3.2), we obtain

$$pe^{i\lambda} \sec \lambda \,.\, b_m + \sum_{k=0}^{m-1} (p - 2\alpha - ip \tan \lambda) a_{p+k} b_{m-k}$$

$$= \{p + ip \tan \lambda - (p + m) e^{i\lambda} \sec \lambda\} a_{p+m};$$

which shows that a_{p+m} on right-hand side depends only on

$$a_p, a_{p+1}, \ldots, a_{p+(m-1)}$$

of left-hand side. Hence for $k \geqslant 0$, we write

$$\sum_{k=0}^{m-1} [\{(p + k) e^{i\lambda} \sec \lambda + (p - 2\alpha - ip \tan \lambda)\} a_{p+k} z^k] \omega(z)$$

$$= \sum_{k=0}^{m} [p + ip \tan \lambda - (p + k) e^{i\lambda} \sec \lambda] a_{p+k} z^k + \sum_{k=m+1}^{\infty} A_k z^k$$

for $m = 1, 2, 3, \ldots$ and a proper choice of A_k.

Let $z = re^{i\theta}$, $0 < r < 1$, $0 \leqslant \theta < 2\pi$, then

$$\sum_{k=0}^{m-1} \mid (p+k) e^{i\lambda} \sec\lambda + (p - 2\alpha - ip\tan\lambda) \mid^2 \mid a_{p+k} \mid^2 r^{2k}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m-1} \{(p+k) e^{i\lambda} \sec\lambda + (p - 2\alpha - ip\tan\lambda) a_{p+k} r^k e^{i\theta k} \right|^2 d\theta$$

$$\geqslant \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m-1} \{(p+k) e^{i\lambda} \sec\lambda + (p - 2\alpha - ip\tan\lambda)\} a_{p+k} r^k e^{i\theta k} \right|^2 \times \mid \omega(re^{i\theta}) \mid^2 d\theta$$

$$\geqslant \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{m} \{p + ip\tan\lambda - (p+k) e^{i\lambda} \sec\lambda\} a_{p+k} r^k e^{i\theta k} + \sum_{k=m+1}^{\infty} A_k r^k e^{ik\theta} \right|^2 d\theta$$

$$\geqslant \sum_{k=0}^{m} \mid p + ip\tan\lambda - (p+k) e^{i\lambda} \sec\lambda \mid^2 \mid a_{p+k} \mid^2 r^{2k} + \sum_{k=m+1}^{\infty} \mid A_k \mid^2 r^{2k}$$

$$\geqslant \sum_{k=0}^{m} \mid p + ip\tan\lambda - (p+k) e^{i\lambda} \sec\lambda \mid^2 \mid a_{p+k} \mid^2 r^{2k}. \qquad \ldots(3.3)$$

Setting $r \to 1$ in (3.3), the inequality (3.3) may be written as

$$\sum_{k=0}^{m-1} \left\{ \begin{array}{l} \mid (p+k) e^{i\lambda} \sec\lambda + (p - 2\alpha - ip\tan\lambda) \mid^2 \\ \quad - \mid p + ip\tan\lambda - (p+k) e^{i\lambda} \sec\lambda \mid^2 \end{array} \right\} \mid a_{p+k} \mid^2$$
$$\geqslant \mid p + ip\tan\lambda - (p+m) e^{i\lambda} \sec\lambda \mid^2 \mid a_{p+m} \mid^2. \qquad \ldots(3.4)$$

Simplification of (3.4) leads to

$$\mid a_{p+m} \mid^2 \leqslant \frac{4(p-\alpha)\cos^2\lambda}{m^2} \sum_{k=0}^{m-1} (p+k-\alpha) \mid a_{p+k} \mid^2. \qquad \ldots(3.5)$$

Replacing $p+m$ by n in (3.5), we are led to

$$\mid a_n \mid^2 \leqslant \frac{4(p-\alpha)\cos^2\lambda}{(n-p)^2} \sum_{k=0}^{n-(p+1)} (p+k-\alpha) \mid a_{p+k} \mid^2. \qquad \ldots(3.6)$$

where $\quad n \geqslant p+1.$

For $n = p + 1$, (3.6) reduces to

$$| a_{p+1} |^2 \leqslant 4(p - \alpha)^2 \cos^2 \lambda$$

or

$$| a_{p+1} | \leqslant 2(p - \alpha) \cos \lambda \qquad \ldots(3.7)$$

which is equivalent to (3.1).

To establish (3.1) for $n > p + 1$, we will apply induction argument.

Fix n, $n \geqslant p + 2$, and suppose (3.1) holds for $k = 1, 2, \ldots, n - (p + 1)$. Then

$$| a_n |^2 \leqslant \frac{4(p - \alpha) \cos^2 \lambda}{(n - p)^2} \times \left\{ (p - \alpha) + \sum_{k=1}^{n-(p+1)} (p + k - \alpha) \prod_{j=0}^{k-1} \frac{| 2(p - \alpha) \cos \lambda \, e^{i\lambda} + j |^2}{(j + 1)^2} \right\}. \qquad \ldots(3.8)$$

Thus from (3.6), (3.8) and Lemma 2.5 with $m = n - p$, we obtain

$$| a_n |^2 \leqslant \prod_{j=0}^{n-(p+1)} \frac{| 2(p - \alpha) \cos \lambda e^{-i\lambda} + j |^2}{(j + 1)^2} .$$

This completes the proof of (3.1). This proof is based on a technique found in Clunie (1959).

For sharpness of (3.1) consider

$$F_{(p,\alpha,\lambda)}(z) = \frac{z^p}{(1 - z)^{2(p - \alpha) \cos \lambda \, e^{-i\lambda}}}$$

$$= z^p + \sum_{n=p+1}^{\infty} \prod_{k=0}^{n-(p+1)} \frac{| 2(p - \alpha) \cos \lambda \, e^{-i\lambda} + k |}{k + 1} z^n. \qquad \ldots(3.9)$$

Setting $p = 1$, in Theorem 3.1, we get Theorem 1 of Libera (1967) which is stated in the following corollary.

Corollary 1 (Libera 1967) — If $f \in S^\lambda(1, \alpha) \equiv S^\lambda(\alpha)$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E \qquad \ldots(3.10)$$

then

$$| a_n | \leqslant \prod_{k=0}^{n-2} \frac{| 2(1 - \alpha) \cos \lambda \, e^{-i\lambda} + k |}{k + 1}, \quad n = 2, 3, \ldots, \qquad \ldots(3.11)$$

and these bounds are sharp for all admissible λ and α and for each n.

Choosing $p = 1$ and $\alpha = 0$, we get the result due to Zamorski (1962) as following corollary.

Corollary 2 (Zamorski 1962) — If $f \in S^{\lambda}(1, 0) \equiv S^{\lambda}$ and has the representation (3.10), then

$$| a_n | \leqslant \prod_{k=0}^{n-2} \frac{| 2e^{-i\lambda} \cos \lambda + k |}{k + 1}, \quad n = 2, 3, \ldots . \qquad \ldots(3.12)$$

Taking $p = 1$ and $\lambda = 0$, we obtain a theorem of Robertson (1936) which was also obtained by Schild (1965) as following corollary.

Corollary 3 (Robertson 1936, Schild 1965) — If f is starlike of order α and has (3.10) as its Maclaurin series, then

$$| a_n | \leqslant \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n - 1)!}, \quad n = 2, 3, \ldots . \qquad \ldots(3.13)$$

By setting $p = 1$, $\alpha = 0$ and $\lambda = 0$ in (3.6) of preceding theorem, we get a result established by Clunie and Keogh (1960) and also by Pommerenke (1962).

Corollary 4 — If f is starlike and has the representation (3.10), then

$$(n - 1)^2 | a_n |^2 \leqslant 4 \left[1 + \sum_{k=2}^{n-1} k | a_k |^2 \right], \quad n = 2, 3, \ldots . \qquad \ldots(3.14)$$

Theorem 3.2 — If $f \in S^{\lambda}(p, \alpha)$, then (a) for any real number μ, we have

$$| a_{p+2} - \mu a_{p+1}^2 | \leqslant \begin{cases} (p - \alpha) \cos \lambda \left[\cos \lambda \{2(p - \alpha)(1 - 2\mu) + 1\} + | \sin \lambda | \right] & \text{if } \mu \leqslant \frac{1}{2} \\ (p - \alpha) \cos \lambda [\cos \lambda + | \sin \lambda |] & \text{if } \frac{1}{2} \leqslant \mu \leqslant \dfrac{1 + p - \alpha}{2(p - \alpha)} \\ (p - \alpha) \cos \lambda \left[\cos \lambda \{2(p - \alpha)(2\mu - 1) - 1\} + | \sin \lambda | \right] & \text{if } \mu \geqslant \dfrac{1 + p - \alpha}{2(p - \alpha)}. \end{cases} \qquad \ldots(3.15)$$

and (b) for any complex number μ, we have

$$| a_{p+2} - \mu a_{p+1}^2 | \leqslant (p - \alpha) \cos \lambda \times \max (1, | 2(p - \alpha)(2\mu - 1) \cos \lambda - e^{i\lambda} |). \qquad \ldots(3.16)$$

The result is sharp for each μ either real or complex.

PROOF : As $f \in S^{\lambda}(p, \alpha)$, from Lemma 2.4 we have

$$e^{i\lambda} \sec \lambda \frac{zf'(z)}{f(z)} - ip \tan \lambda = \frac{p - (p - 2\alpha)\, \omega(z)}{1 + \omega(z)} \quad \ldots(3.17)$$

where $\omega(z) = \sum_{k=1}^{\infty} b_k z^k \in B.$

Rewriting the form (3.17) as

$$\omega(z) = \frac{p - e^{i\lambda} \sec \lambda\, (zf'(z)/f(z)) + ip \tan \lambda}{e^{i\lambda} \sec \lambda\, (zf'(z)/f(z)) - ip \tan \lambda + (p - 2\alpha)}$$

$$= \frac{e^{i\lambda} \sec \lambda\, \{pf(z) - zf'(z)\}}{e^{i\lambda} \sec \lambda\, (zf'(z) + \{pe^{-i\lambda} \sec \lambda - 2\alpha\}\, f(z)}$$

$$= \frac{-\, e^{i\lambda} \sec \lambda \sum_{k=1}^{\infty} k a_{p+k} z^k}{2(p - \alpha) \left\{1 + \sum_{k=1}^{\infty} a_{p+k} z^k\right\} + e^{i\lambda} \sec \lambda \sum_{k=1}^{\infty} k a_{p+k} z^k}$$

$$= \frac{-\, e^{i\lambda} \sec \lambda \sum_{k=1}^{\infty} k a_{p+k} z^k}{2(p - \alpha) + \sum_{k=1}^{\infty} [\{2(p - \alpha) + k e^{i\lambda} \sec \lambda\}\, a_{p+k}]\, z^k}$$

$$= -\, e^{i\lambda} \sec \lambda \left[\frac{a_{p+1}}{2(p - \alpha)} z + \frac{1}{2(p - \alpha)} \times \left\{2a_{p+2} - \left(\frac{2(p - \alpha) + e^{i\lambda} \sec \lambda}{2(p - \alpha)}\right) a_{p+1}^2\right\} z^2 + \ldots\right]$$

and then comparing coefficients of z and z^2 on both sides, we have

$$b_1 = -\frac{e^{i\lambda} \sec \lambda}{2(p - \alpha)} a_{p+1}$$

$$b_2 = -\frac{e^{i\lambda} \sec \lambda}{4(p - \alpha)^2} [4(p - \alpha)\, a_{p+2} - \{2(p - \alpha) + e^{i\lambda} \sec \lambda\}\, a_{p+1}^2].$$

Thus

$$a_{p+1} = -\frac{2(p - \alpha)}{e^{i\lambda} \sec \lambda} b_1$$

and

$$a_{p+2} = -\frac{(p - \alpha)}{e^{i\lambda} \sec \lambda} b_2 + \frac{2(p - \alpha) + e^{i\lambda} \sec \lambda}{4(p - \alpha)} a_{p+1}^2,$$

Hence

$$a_{p+2} - \mu a_{p+1}^2$$
$$= -\frac{(p-\alpha)}{e^{i\lambda}\sec\lambda} b_2 + \left[\frac{2(p-\alpha)+e^{i\lambda}\sec\lambda}{4(p-\alpha)} - \mu\right] a_{p+1}^2$$
$$= -\frac{(p-\alpha)}{e^{i\lambda}\sec\lambda} b_2 + \left[\frac{2(p-\alpha)+e^{i\lambda}\sec\lambda}{4(p-\alpha)} - \mu\right] \frac{4(p-\alpha)^2}{e^{2i\lambda}\sec^2\lambda} b_1^2. \qquad ...(3.18)$$

Thus taking modulus of both sides of (3.18), we are led to

$$| a_{p+2} - \mu a_{p+1}^2 |$$
$$= \frac{p-\alpha}{\sec\lambda}\left| b_2 - \left\{\frac{2(p-\alpha)+e^{i\lambda}\sec\lambda}{4(p-\alpha)} - \mu\right\} \frac{4(p-\alpha)}{e^{i\lambda}\sec\lambda} b_1^2\right|. \qquad ...(3.19)$$

(a) *When μ is Real*

For real μ, (3.19) becomes

$$| a_{p+2} - \mu a_{p+1}^2 |$$
$$\leqslant \frac{p-\alpha}{\sec\lambda} [\, | b_2 | + | 2(p-\alpha)(2\mu - 1)\cos\lambda - e^{i\lambda} | \; | b_1 |^2]. \qquad ...(3.20)$$

Applying Lemma 2.1 for $| b_2 |$ in (3.20) we obtain

$$| a_{p+2} - \mu a_{p+1}^2 |$$
$$\leqslant (p-\alpha)\cos\lambda\,[1 + \{\, | 2(p-\alpha)(2\mu-1)\cos\lambda - e^{i\lambda} | - 1\} \, | b_1 |^2]. \qquad ...(3.21)$$

Again using Lemma 2.1 for $| b_1 |$ in (3.21) we are led to

$$| a_{p+2} - \mu a_{p+1}^2 |$$
$$\leqslant (p-\alpha)\cos\lambda \, | 2(p-\alpha)(2\mu-1)\cos\lambda - e^{i\lambda} |$$
$$= (p-\alpha)\cos\lambda\,[\cos\lambda \, . \, | 2(p-\alpha)(2\mu-1) - 1 | + | \sin\lambda | \,] \qquad ...(3.22)$$

Thus from (3.22) with simple computations we obtain the results of (3.15) stated in (a) of the theorem for various values of real μ.

(b) *When μ is a Complex Number*

For any complex number μ (3.19) may be written as

$$| a_{p+2} - \mu a_{p+1}^2 |$$
$$\leqslant (p-\alpha)\cos\lambda \, . \left| b_2 - \left\{\frac{e^{i\lambda}\sec\lambda - 2(p-\alpha)(2\mu-1)}{e^{i\lambda}\sec\lambda}\right\} b_1^2 \right|. \qquad ...(3.23)$$

Using Lemma 2.2 in (3.23) we get

$$| a_{p+2} - \mu a_{p+1}^2 |$$
$$\leqslant (p - \alpha) \cos \lambda \,.\, \max \left(1, \left| \frac{2(p - \alpha)(2\mu - 1) - e^{i\lambda} \sec \lambda}{e^{i\lambda} \sec \lambda} \right| \right)$$
$$= (p - \alpha) \cos \lambda \,.\, \max (1, | 2(p - \alpha)(2\mu - 1) \cos \lambda - e^{i\lambda} |) \qquad \ldots(3.24)$$

which is (3.16) in (b) of the theorem.

The sharpness of (3.15) and (3.16) follows from that of (2.2).

Setting $p = 1$ in Theorem 3.2, we get the result for univalent λ-spiral function of order α, as stated in the following corollary

Corollary 5 — If $f \in S^{\lambda}(1, \alpha) \equiv S^{\lambda}(\alpha)$, then (a) for any real number μ, we have

$$| a_3 - \mu a_2^2 |$$
$$\leqslant \begin{cases} (1 - \alpha) \cos \lambda \,[\cos \lambda \,\{2(1 - \alpha)(1 - 2\mu) + 1\} + | \sin \lambda |] & \text{if } \mu \leqslant \frac{1}{2} \\ (1 - \alpha) \cos \lambda \,[\cos \lambda + | \sin \lambda |] & \text{if } \frac{1}{2} \leqslant \mu \leqslant \frac{2 - \alpha}{2(1 - \alpha)} \\ (1 - \alpha) \cos \lambda \,[\cos \lambda \,\{2(1 - \alpha)(2\mu - 1) - 1\} + | \sin \lambda |] & \text{if } \mu \geqslant \frac{2 - \alpha}{2(1 - \alpha)} \end{cases} \qquad \ldots(3.25)$$

and (b) for any complex number μ, we have

$$| a_3 - \mu a_2^2 |$$
$$\leqslant (1 - \alpha) \cos \lambda \,.\, \max (1, | 2(1 - \alpha)(2\mu - 1) \cos \lambda - e^{i\lambda} |). \qquad \ldots(3.26)$$

The result is sharp.

Putting $p = 1$ and $\alpha = 0$, we get the result for univalent λ-spiral function as follows.

Corollary 6 — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^{\lambda}(1, 0) \equiv S^{\lambda}$, then (a) for any real number μ, we have

$$| a_3 - \mu a_2^2 |$$
$$\leqslant \begin{cases} \cos \lambda \,[\cos \lambda \,(3 - 4\mu) + | \sin \lambda |] & \text{if } \mu \leqslant \frac{1}{2} \\ \cos \lambda \,[\cos \lambda + | \sin \lambda |] & \text{if } \frac{1}{2} \leqslant \mu \leqslant 1 \\ \cos \lambda \,[\cos \lambda \,(4\mu - 3) + | \sin \lambda |] & \text{if } \mu \geqslant 1 \end{cases} \qquad \ldots(3.27)$$

and (b) for any complex number μ, we obtain

$$| a_3 - \mu a_2^2 | \leqslant \cos \lambda \,.\, \max (1, | 2(2\mu - 1) \cos \lambda - e^{i\lambda} |). \qquad ...(3.28)$$

The result is sharp.

Choosing $\lambda = 0$, in Theorem 3.2 we have the result for p-valent starlike function of order α as follows.

Corollary 7 — If $f \in S^*(p, \alpha)$, then (a) for any real number μ, we get

$$| a_{p+2} - \mu a_{p+1}^2 | \leqslant \begin{cases} (p - \alpha) [2(p - \alpha) (1 - 2\mu) + 1] & \text{if } \mu \leqslant \frac{1}{2} \\ p - \alpha & \text{if } \frac{1}{2} \leqslant \mu \leqslant \dfrac{1 + p - \alpha}{2(p - \alpha)} \\ (p - \alpha) [2(p - \alpha) (2\mu - 1) - 1] & \text{if } \mu \geqslant \dfrac{1 + p - \alpha}{2(p - \alpha)} \end{cases} \qquad ...(3.29)$$

and (b) for any complex number μ, we have

$$| a_{p+2} - \mu a_{p+1}^2 | \leqslant (p - \alpha) \max (1, | 2(p - \alpha) (2\mu - 1) - 1 |). \qquad ...(3.30)$$

The result is sharp.

Remarks

(1) Setting $p = 1$ in Corollary 7, we obtain the result for univalent Starlike function of order α.

(2) Further putting $p = 1$ and $\alpha = 0$ in Corollary 7, we get the result for univalent Starlike function found in Singh and Singh (1974) and also in Keogh and Merkes (1969) for complex number μ only.

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