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POSITIVE DEFINITE MATRICES AND ABSOLUTELY MONOTONIC FUNCTIONS

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Let $\mathcal{M}_n$ denote the class of all functions $f(x)$ defined on $x > 0$ such that if $A = (a_{ij})$ is a matrix of order n, $a_{ij} > 0$ and A is a positive definite matrix, then $B = (f(a_{ij}))$ has the same two properties. We characterise the class $\mathcal{M}_2$ and $\mathcal{M}_\infty = \bigcap_{n=1}^{\infty} \mathcal{M}_n$.

Let $f(x)$ be a real-valued function defined on $x > 0$. Consider the following properties which $f(x)$ may satisfy :

(I) If $a_{ij} > 0$ and if $A = (a_{ij})$ is a positive definite matrix of any order, then the matrix $(f(a_{ij}))$ is also positive definite with positive entries.

(II) $\quad f(x) = \sum_{n=0}^{\infty} a_n x^n (x > 0)$ with $a_n \geqslant 0$, $n = 0, 1, 2, \ldots$

In what follows, we shall show that a function which satisfies (I) is of the form (II) and conversely.

Let us recall that $A = (a_{ij})$ is said to be a positive definite matrix of order n if

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant 0$$

for every real vector $(\xi_1, \xi_2, \ldots, \xi_n)$. A function f is said to be absolutely monotonic in (a, b) if it is infinitely differentiable in (a, b) and if all its derivatives are non-negative there; one can also characterise absolutely monotonic functions by means of higher differences, without any continuity assumptions (Widder 1941).

Many authors (Herz 1962, Rudin 1959, Schoenberg 1942) have considered a similar problem. Although both the assumptions and conclusions are of a similar nature yet the proofs of the Theorems 1 and 6 are quite different, much simpler and more direct.

Define $\mathcal{M}_n$ to be the class of all functions $f(x)$ defined on $x > 0$ such that if $A = (a_{ij})$ is a matrix of order n, $a_{ij} > 0$ and A is a positive definite matrix, then

$B = (f(a_{ij}))$ has the same two properties. Our problem is then to characterise the class $\mathcal{M}_\infty = \bigcap_{n=1}^{\infty} \mathcal{M}_n$.

It is evident that if $f \in \mathcal{M}_n$, then $f(a_{ij}) > 0$ for every a_{ij}, that is, f is a positive function. Moreover, $f^2(a_{ij}) \leqslant f(a_{ii})\, f(a_{jj})$. The class of functions $\mathcal{M}_n$ is a convex cone, closed under pointwise limits and contains the functions $f(x) = x$ and $f(x) = k > 0$. From Schur's theorem [If (a_{ij}) and (b_{ij}) are positive definite matrices of the same order, then so is the matrix (c_{ij}), where $c_{ij} = a_{ij}b_{ij}$] it follows that the product of any two functions in $\mathcal{M}_n$ is again in $\mathcal{M}_n$. Consequently,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \; a_n \geqslant 0,$$

defined on $x > 0$ belongs to $\mathcal{M}_n$.

We have thus proved the following :

Theorem 1 — If f is of the form II, then f satisfies I.

Our next result characterise the class $\mathcal{M}_2$.

Theorem 2 — Let f be a real function defied on $x > 0$. Then $f \in \mathcal{M}_2$ iff $g(x) = \log f(e^x)$ is midpoint convex and f is monotonically increasing. In particular, if $f \in \mathcal{M}_2$, then f is continuous.

PROOF : Suppose $f \in \mathcal{M}_2$. Then $\sqrt{f(a)\, f(c)} \geqslant f(\sqrt{ac})$ where $a > 0$ and $c > 0$. That is,

$$\tfrac{1}{2}\{\log f(a) + \log f(c)\} \geqslant \log f(\sqrt{ac}).$$

Thus

$$g\left(\frac{\log a + \log c}{2}\right) \leqslant \frac{g(\log a) + g(\log c)}{2}$$

that is, $g(x) = \log f(e^x)$ is midpoint convex. Moreover, $f(\sqrt{ac}) \geqslant f(b)$, where $b^2 \leqslant ac(a, b, c \geqslant 0)$, implies that f is monotonically increasing.

On the other hand $f(a)\, f(c) = e^{g(\log a) + g(\log c)}$

$$\geqslant e^{2g((\log a + \log c)/2)}$$

iff $$\frac{g(\log a) + g(\log c)}{2} \geqslant g\left(\frac{\log a + \log c}{2}\right),$$

that is, $$\frac{\log f(a) + \log f(c)}{2} \geqslant \log f(\sqrt{ac}).$$

This implies $\sqrt{f(a)\, f(c)} \geqslant f(\sqrt{ac}) \geqslant f(b)$, using monotonicity of f. Thus

$$f(a)\, f(c) \geqslant f^2(b),$$

where a, b, c are positive and $ac \geqslant b^2$.

Lemma 3 — Let $f : (a-\epsilon, a+\epsilon) \to \mathbf{R}$ be a C^{n-1} map satisfying the following : For any distinct positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ let

$$\sum_{i,j=1}^{n} \beta_i \beta_j f(a + t\alpha_i\alpha_j) \geqslant 0 \qquad \ldots(1)$$

for all real numbers $\beta_1, \beta_2, \ldots, \beta_n$ and sufficiently small t. Then $f^{(j)}(a) \geqslant 0$ for all $j = 0, 1, 2, \ldots, n-1$.

PROOF : Let $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$

$$\underline{\alpha}^{(j)} = (\alpha_1^j, \alpha_2^j, \ldots, \alpha_n^j) \quad j = 0, 1, 2, \ldots, n-1.$$

Note that $\underline{\alpha}^{(j)}$, $j = 0, 1, 2, \ldots, n-1$ are n linearly independent vectors in $\mathbf{R}^n$. Choose $\underline{\beta}$ orthogonal to $\underline{\alpha}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(k-1)}$ but not to $\underline{\alpha}^{(k)}$. Here $k \leqslant n-1$. In (1) expanding f by Taylor series around a, we get

$$\sum_{i,j} \beta_i \beta_j \left[\sum_{r=0}^{k=1} \frac{(t\alpha_i\alpha_j)^r}{r!} f^{r}(a) + \frac{(t\alpha_i\alpha_j)^k}{k!} f^{(k)}(a + \theta_{ij} t\alpha_i\alpha_j) \right] \geqslant 0$$

where $0 < \theta_{ij} < 1$. Since $\underline{\beta} \cdot \underline{\alpha}^{(r)} = 0$ for $r \leqslant k-1$, we obtain

$$\frac{t^k}{k!} \sum_{i,j} \beta_i \beta_j \alpha_i^k \alpha_j^k f^{(k)}(a + \theta_{ij} t\alpha_i\alpha_j) \geqslant 0.$$

Cancelling $t^k/k!$ and letting $t \to 0$, we obtain

$$f^{(k)}(a) \geqslant 0.$$

Lemma 4 — Let f be a continuous function defined on $\{x \in \mathbf{R} : x \geqslant 0\}$. Assume that for each $a > 0$, $n = 1, 2, 3, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 0$,

$$\sum_{i,j} \beta_i \beta_j f(a + t\alpha_i\alpha_j) \geqslant 0 \qquad \ldots(2)$$

for all t sufficiently small. Let ϕ be any C^∞ probability density function with compact support in $(0, \infty)$. Let

$$f_\phi(x) = \int_0^\infty f(xy^{-1})\, \phi(y) \frac{dy}{y}. \qquad \ldots(3)$$

Then f_ϕ is C^∞ in $[0, \infty)$ and $f_\phi^{(j)}(x) \geqslant 0$ for all $x \in [0, \infty)$. In particular, by a result of Bernstein,

$$f_\phi(x) = \sum_0^\infty P_j(\phi)\, x^j \quad \text{for all } x \in [0, \infty)$$

where $P_j(\phi) \geqslant 0$ and

$$\sum P_j(\phi) = \int_0^\infty f(y^{-1})\, \phi(y) \frac{dy}{y}.$$

PROOF: It is clear from (3) that $f_\phi \in C^\infty([0, \infty))$. Further (2) implies that f_ϕ satisfies Lemma 3. Hence $f_\phi^{(j)}(x) \geqslant 0$ for all $x \geqslant 0$. The last two parts are consequences of Bernstein's theorem.

Corollary 5 — If f satisfies the conditions of Lemma 4, then

$$f(x) = \sum_0^\infty P_j x^j \quad \text{for all } x \in [0, \infty)$$

where $P_j \geqslant 0$ for every j.

PROOF: This is immediate from Lemma 4 by letting the probability distribution corresponding to ϕ converge weakly to the distribution degenerate at 1.

Theorem 6 — If $f \in \mathcal{M}_\infty$ then $f(x) = \sum_{i=0}^\infty a_j x_j$, for all $x \in [0, \infty)$, where $a_j \geqslant 0$ for all j.

PROOF: This is immediate from Theorem 2 and corollary 5.

Remark: It will be of interest to know the classes $\mathcal{M}_k$, $3 \leqslant k \leqslant \infty$. It is natural to expect that the classes $\mathcal{M}_k$ will become more specialised with increasing k. In fact, the function

$$f(x) = \begin{cases} x & \text{if } x \leqslant 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

does not belong to $\mathcal{M}_3$. One need only consider the matrix

$$\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 4 & 1 \\ \frac{1}{2} & 1 & 4 \end{pmatrix}.$$

However, the function f does belong to $\mathcal{M}_2$.

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