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# MIXED SEMIGROUPS OF OPERATORS ON A LOCALLY CONVEX SPACE

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Mixed semigroups of linear operators are considered on a locally convex Hausdorff linear topological space. The results obtained in this direction extend to those of Singh (1978). Examples of mixed semigroups are also given.

Let $E$ denote a locally convex Hausdorff linear topological space. Let $L(E, E)$ denote the space of continuous linear operators on $E$. Buche and Vasudeva (1971) introduced the generalized Cauchy equation satisfied by operators in $L(X, X)$ where $X$ is a Banach Space. In the present work we introduce the generalized Cauchy equation in $L(E, E)$, and study a particular case of this which we call by Mixed Semigroup of operators in $L(E, E)$.

Mainly the results obtained in the present paper extend to those of Singh (1978) when the underlying space is a locally convex Hausdorff linear topological space. In section 1 we give some definitions and preliminary results. In sections 2-4 we give certain representation and generation theorems for mixed semigroups.

§1. Let $R^+ = [0, \infty)$. Let $L_s(E, E)$ be the space $L(E, E)$ mentioned in the introduction furnished with the topology of simple convergence and $L_C(E, E)$ is the space $L(E, E)$ furnished with the topology of uniform convergence on the system of all bounded subsets of $E$ denoted by $C$, (see Bourbaki 1955, Schaeffer 1966). The family $\{S(t), t \in R^+\}$, $S : R^+ \to L(E, E)$ is said to satisfy a generalized Cauchy equation if

$$S(t + s) = H(S(s), S(t)), \ s, t \in R^+ \qquad ...(1)$$

where $H : L(E, E) \times L(E, E) \to L(E, E)$ is a function (operator valued). In particular when $H(S(s), S(t)) = S(s) S(t)$ then above family $\{S(t), t \in R^+\}$ reduces to a semigroup.

We now specialize the function $H$ to be of the following type.

$$H(S(s), S(t)) = S(s) S(t) + \alpha(S(s) - T(s)) (S(t) - T(t)) \qquad ...(2)$$

where $S(0) = I$ and $\alpha$ is any real number, and $T(t)$ is a known semigroup of operators belonging to $L(E, E)$. When $\alpha = 0$, $\{S(t)\}$ again reduces to an usual semigroup. Eventually we will deal with two different cases $\alpha \neq -1$ and $\alpha = -1$.

Next we state some known results for semigroups in a locally convex space.

A semigroup $\{T(t), t \in R^+\}$ is said to be continuous if for $x \in E$, the function $t \to T(t)\, x$ is continuous from $(0, \infty)$ to $E$. The operator $A_0$ on $E$ defined by

$$A_0 x = \lim_{t \to 0} \frac{T(t)\, x - x}{t}$$

for each $x \in E$ at which the limit exists is called the infinitesimal generator of the semigroup $\{T(t), t \in R^+\}$. It is clear that the domain $\mathcal{D}(A_0)$ of $A_0$ is a subspace of $E$ and $A_0$ is a linear operator from $\mathcal{D}(A_0)$ to $E$. It can also be verified that the range of $A_0$ is contained in the closure of $\bigcup_{t>0} T(t)\, E = E_0$ (say). For a continuous semigroup $\{T(t), t \in R^+\}$ we have also the fact that for $x \in E$ and $[\alpha, \beta] \subset [0, \infty)$, $\int_\alpha^\beta T(t)\, x dt \in E$ if for instance $E$ is quasi-complete (see Singbal-Vedak 1972). The following theorem (see Singbal-Vedak 1972) gives the denseness of $\mathcal{D}(A_0)$ in $E_0$.

*Theorem* 1.1 — Let $T(t)$ be a continuous semigroup of operators on $E$ which is quasi-complete. Then it follows that (i) $\mathcal{D}(A_0)$ is dense in $E_0$ and (ii) $\mathcal{D}(A_0)$ and $E_0$ have the same closure.

§2. Let $\{S(t), t \in R^+\}$ be a continuous family of operators in $L(E, E)$ which satisfy (1) with the function $H$ as in (2) with $\alpha \neq -1$. Let $A_0$ be the infinitesimal generator of $\{T(t)\}$. We define the first infinitesimal generator $A$ of $\{S(t)\}$ as

$$Ax = \lim_{t \to 0+} \frac{S(t) - I}{t}\, x$$

for $x \in \mathcal{D}(A)$, $\mathcal{D}(A)$ being the set of elements $x \in E$ for which the above limit exists. Following lemma of Singh (1978) can be seen to be equally valid for operators in $L(E, E)$ so we omit the proof of the lemma.

*Lemma* 2.1 — Let $\{S(t), t \in R^+\}$ be a continuous family of mixed semigroups with $\alpha \neq -1$. Let $T_1(t)\, x = (1 + \alpha)\, S(t)\, x - \alpha T(t)\, x$, for $x \in E$, then $T_1(t)$ is a continuous semigroup.

*Lemma* 2.2 — Set $A_1 = (1 + \alpha)\, A - \alpha A_0$. Then $A_1$ is the infinitesimal generator of the semigroup $T_1(t)$.

PROOF : We have

$$\lim_{s \to 0+} \frac{T(s)\, x - x}{s} = \lim_{s \to 0+} \frac{(1 + \alpha)\, S(s)\, x - \alpha T(s)\, x - x}{s}$$

(*continued on p.* 861)

$$= \lim_{s \to 0+} \left\{ (1 + \alpha) \frac{(S(s) x - x)}{s} - \alpha \frac{T(s) x - x}{s} \right\}$$

$$= (1 + \alpha) Ax - \alpha A_0 x = A_1 x, \text{ provided } x \in \mathscr{D}(A_1).$$

Next we discuss about the respective domains of $A_1$, $A$, and $A_0$ in the following paragraph.

If we denote $E_1 = \bigcup_{s>0} T_1(s) x$ for $x \in E$ and $F_0 = \bigcup_{s>0} T(s) x$ for $x \in E$, then from Theorem 1.1 it follows that $\mathscr{D}(A_0)$ and $\mathscr{D}(A_1)$ are dense in $F_0$ and $E_1$ respectively. We define $\mathscr{D}(A)$ to be equal to $\mathscr{D}(A_1) \cap \mathscr{D}(A_0)$ and we can only infer that $\overline{\mathscr{D}(A)} = \overline{\mathscr{D}(A_1) \cap \mathscr{D}(A_0)} \subseteq \overline{\mathscr{D}(A_1)} \cap \overline{\mathscr{D}(A_0)} = E_1 \cap F_0$.

Following theorem gives the representation of $\{S(t)\}$ in $L_s(E, E)$, which follows easily from the above considerations and Lemmas 2.1 and 2.2.

*Theorem* 2.3 — Let $S(t)$ be a continuous family of mixed semigroups of operators. Then for $\alpha \neq -1$

$$S(t)\, x = (\alpha/(1 + \alpha))\, T(t)\, x + (1/(1 + \alpha))\, T_1(t)\, x \quad \text{for } x \in E$$

where the operators of both the sides coincide in $L_s(E, E)$.

§3. A semigroup of operators $\{T(t), t \in R^+\}$ on $E$ is said to be equicontinuous if $\{T(t)\}_{t>0}$ is an equicontinuous family of operators on $E$, i.e., for any continuous semi-norm $p$ on $E$, there exists a continuous semi-norm $q$ on $E$ such that $p(T(t)\, x) \leqslant q(x)$ for all $t \geqslant 0$ and all $x \in E$. $\{T(t)\}_{t>0}$ is said to be locally equicontinuous if for every $[\alpha, \beta] \subset (0, \infty)$ the family $\{T(s)\}_{\alpha \leqslant s \leqslant \beta}$ is equicontinuous. If $A_0$ denotes the infinitesimal generator of an equicontinuous semigroup then it can be shown as in Theorem 1.1 (see also Yosida 1965) that $\mathscr{D}(A_0)$ is dense in $E$. Following theorem is usually known as the Hille-Yosida theorem, which gives necessary and sufficient conditions in order that a closed linear densely defined operator be the infinitesimal generator of a continuous and equicontinuous semigroup. For the proof see Yosida (1965) (also see Schwartz 1958).

*Theorem* 3.1 — Let $A$ be a closed linear densely defined operator on a sequentially complete locally convex Hausdorff linear topological space $E$. Then in order that $A$ be the infinitesimal generator of a continuous and equicontinuous semigroup $\{T(s)\}_{s>0}$ it is necessary and sufficient that the family

$$\{\lambda^n [R(\lambda, A)]^n : \lambda > 0,\ n = 1, 2, 3, \ldots\}$$

is equicontinuous, where $R(\lambda, A)$ denotes the resolvant operator $(\lambda I - A)^{-1}$ of $A$.

Next we give a Hille-Yosida type of theorem for mixed semigroups.

*Theorem* 3.2 — Let $A$ be a linear operator on a locally convex sequentially complete Hausdorff linear topological space $E$. Then a necessary and sufficient condition that $A$ be the first infinitesimal generator of a one-parameter family $\{S(t)\}$ of operators satisfying (1) with $H$ as in (2) with $\alpha \neq -1$ and with $\{e^{-ws}T_1(s)\}$ equicontinuous for some $w > 0$, is that $A_1 = (1 + \alpha) A - \alpha A_0$ is a closed densely defined operator on $E$ and the family $\{(\lambda - w)^n [R(\lambda, A)]^n : \lambda > w, n = 1, 2, 3, ...\}$ is equicontinuous.

PROOF : *Necessity* : In fact by Lemma 2.1 and 2.2, $T_1(t)$ is a continuous semigroup with generator $A_1$. Also if $\{e^{-tw}T_1(t)\}_{t\geqslant 0}$ is equicontinuous and $B$ be its infinitesimal generator, then denoting $U(t) = e^{-wt}T_1(t)$ we get for $x \in \mathcal{D}(A_1)$,

$$\frac{U(t)\,x - x}{t} = (\exp(-wt) - 1)/t\; T_1(t)\,x + \frac{1}{t}\,(T_1(t)\,x - x).$$

Hence $Bx$ exists and $Bx = Ax - wx$. Thus $\mathcal{D}(B) \subset \mathcal{D}(A_1)$. On the other hand $T_1(s) = e^{ws}U(s)$ so that by the same argument as above we have $\mathcal{D}(A_1) \subset \mathcal{D}(B)$ and we get $\mathcal{D}(A_1) = \mathcal{D}(B)$ and $B = A_1 - wI$. Since $B$ is closed therefore $A_1$ is also closed. Since $B$ is the infinitesimal generator of an equicontinuous semigroup, therefore, from the discussions preceding Theorem 3.1 it follows that $\mathcal{D}(B)$ is dense in $E$ and hence $\mathcal{D}(A_1)$ is dense in $E$. Now by Theorem 3.1 we find that the family $\{\mu^n [R(\mu, B)]^n : \mu > 0, n = 1, 2, 3 ...\}_{(*)}$ is equicontinuous. Moreover,

$$\mu I - B = (\mu + w)\, I - A_1.$$

So that $R(\mu, B) = R(\mu + w, A_1)$. Hence by setting $\lambda = \mu + w$ in (*) above we get the equicontinuity of the family $\{(\lambda - w)^n R(\lambda, A_1)^n : \lambda > w, n = 1, 2 ...\}$.

*Sufficiency* : Let $A_1$ be a closed densely defined operator such that the family

$$\{(\lambda - w)^n [R(\lambda, A_1)]^n : \lambda > w,\ n = 1, 2, ...\} \qquad ...(**)$$

is equicontinuous. Then it follows that $B = -wI + A_1$ is also closed and densely defined. Also $R(\lambda, A_1) = R(\lambda - w, B)$. Thus

$$\{\mu^n [R(\mu, B)]^n : \mu > 0,\ n = 1, 2, 3, ...\}$$

is equicontinuous (where $\mu = \lambda - w$) from (**). From Theorem 3.1 $B$ is the infinitesimal generator of a continuous and equicontinuous semigroup $\{U(s)\}_{s\geqslant 0}$. Hence $A_1$ is the infinitesimal generator of a continuous semigroup $\{T_1(s)\}_{s\geqslant 0}$ with $T_1(s) = e^{ws}U(s)$ and $e^{-ws}T_1(s)$ is equicontinuous. Now let

$$S(t) = (\alpha/(1 + \alpha))\, T(t) + 1/(1 + \alpha)\, T_1(t), \quad \text{and} \quad \alpha \neq -1.$$

Then $S(t)$ satisfies eqn. (1) with $H$ as in (2). It is now easy to check that $A$ is the first infinitesimal generator of the family $\{S(t)\}$.

§4. In this section we obtain some representation theorem for mixed semigroups when $L(E, E)$ is equipped with the topology of uniform convergence on the system of all bounded subsets $\mathcal{C}$ of $E$ which we denote by $L_{\mathcal{C}}(E, E)$ (sometimes called $L(E, E)$ with $\mathcal{C}$-topology), This replaces the representation theorems obtained for the case of uniform topology on $L(X, X)$ where $X$ is a Banach Space (see Singh 1978, Buche and Vasudeva 1976).

Next we give some definitions and preliminary results which are used later in the present section. These results can be found in Schaeffer (1966).

A locally convex space $E$ is barreled if each subset of $E$ which is radial, convex, circled and closed is a neighbourhood of 0. We note the following results without proof.

(1) Each equicontinuous subset of $L(E, E)$ is bounded for $\mathcal{C}$-topology ($E$ is a Hausdorff linear topological space).

(2) Every simply bounded subset $H$ of $L(E, E)$ (bounded for the topology of simple convergence, i.e., in $L_s(E, E)$) is equicontinuous ($E$ is a locally convex Hausdorff topological space which is barreled).

(3) If $E$ is a barreled locally convex Hausdorff linear topological space which is quasi-complete, then any continuous semigroup $\{T(t)\}_{t\geqslant 0}$ is locally equicontinuous. (This result is due to Singbal-Vedak (1965)).

*Theorem* 4.1 — Let $\{S(s)\}_{s\geqslant 0}$ be a continuous mixed semigroup of operators with $\alpha \neq -1$, on a barreled, quasi-complete locally convex Hausdorff linear topological space $E$ such that for some $t_0 > 0$, $T_1(t_0)\, E \subset \mathcal{D}(A_1)$ and $T(s_0)\, E \subset \mathcal{D}(A_0)$. Then,

$$S^{(1)}(s) = \frac{d}{ds}\, S(s) = (\alpha/(1 + \alpha))\, A_0 T(s) + 1/(1 + \alpha)\, A_1 T_1(s)$$

exists as a continuous linear operator for $s \geqq s_0$ and $s \to S^{(1)}(s)$ from $(s_0, \infty)$ to $L_{\mathcal{C}}(E, E)$ is continuous.

The following lemma is due to Singbal-Vedak (1972).

*Lemma* 4.2 — If $U(s)$ is a continuous and locally equicontinuous semigroup of operators on $E$, then for $x \in \mathcal{D}(B_0)$, where $B_0$ is the infinitesimal generator of $\{U(s)\}_{s>0}$, $(1/t)\,(U(t) - I)\, U(s)\, x = A(t)\, U(s)\, x \to B_0 U(s)\, x = U(s)\, B_0 x$ as $t \to 0+$.

PROOF OF THEOREM 4.1 : We need to show that the mappings $s \to A_0 T(s)$ and $s \to A_1 T_1(s)$ from $(s_0, \infty)$ to $L_{\mathcal{C}}(E, E)$ both are continuous at any $s > s_0$. Since the proof of both the cases are similar we give it for the mapping $s \to A_1 T_1(s)$. For $s > 0$, $t \to A_1(t)\, T_1(s)$ is continuous from $(0, \infty)$ to $L_s(E, E)$ and $s \geqslant s_0$.

$\lim_{t \to 0+} A_1(t)\, T_1(s)$ exists in $L_s(E, E)$, where $A_1(t) = (1/t)\,(T_1(t) - I)$. Hence for $s \geqslant s_0$ we see that $\{A_1(t)\, T_1(s)\}_{0<t\leqslant 1}$ is equicontinuous [this follows because it is simply bounded and $E$ is barreled (see result 2)]. Therefore, an application of lemma 4.2 gives the fact that $A_1 T_1(s) \in L(E, E)$.

Now we will show that $s \to A_1 T_1(s)$ from $(s_0, \infty)$ to $L_C(E, E)$ is continuous at any point $s \in (s_0, \infty)$. Given $\delta > 0$ let $|t| < \delta$. Then it suffices to show that for a subset $C(C \in \mathcal{C})$

$$T_1(s + t)\, x - T(s)\, x \to 0 \quad \text{as} \quad t \to 0 \qquad \ldots(*)$$

uniformly for $x \in C$.

For $x \in E$, $T_1(\tau - s_0)\, x$ is a continuous function of $\tau$ from $[s - \delta, s + \delta]$ to $E$. From the relation $A_1 T_1(\tau) = A_1 T_1(s_0)\, T_1(\tau - s_0)$ and the fact that $A_1 T_1(s_0) \in L(E, E)$ it follows that $\tau \to A_1 T_1(\tau)\, x$ is continuous from $[s - \delta, s + \delta]$ to $E$. Therefore for $|t| < \delta$

$$T_1(s + t)\, x - T_1(s)\, x = \int_s^{s+t} A_1 T_1(\tau)\, x\, d\tau.$$

For any continuous semi-norm $p$ on $E$ we thus have

$$p\,[T_1(s + t)\, x - T_1(s)\, x] \leqslant |t|\; p\,[A_1 T_1(\tau_0)\, x]$$

where $\tau_0$ is some point in the interior of $[s - \delta, s + \delta]$. Moreover since $E$ is barreled it follows that $\{T_1(\tau - s_0)\}_{s-\delta\leqslant\tau\leqslant s+\delta}$ is equicontinuous [result (3)]. Again from the relation $A_1 T_1(\tau) = A_1 T_1(s_0)\, T_1(\tau - s_0)$ and the fact that $A_1 T_1(s_0) \in L(E, E)$ it follows that $\{A_1 T_1(\tau)\}_{s-\delta\leqslant\tau\leqslant s+\delta}$ is equicontinuous. Thus from the result (2), $\{A_1 T_1(\tau)\}_{s-\delta\leqslant\tau\leqslant s+\delta}$ is bounded in $L_C(E, E)$ and hence

$$p\{A_1 T_1(\tau)\, x : x \in C,\; s - \delta \leqslant \tau \leqslant s + \delta\}$$

is a bounded set of real numbers, from which we get

$$p\,[T_1(s + t)\, x - T_1(s)\, x] = O(|t|)$$

uniformly for all $x \in C$ and which completes the proof of (*).

To conclude this section we give an example of mixed semigroup for the case $\alpha \neq -1$ and indicate the type of results one would obtain by the similar techniques as discussed in the previous sections for the case $\alpha = -1$.

The example of a mixed semigroup which we give below in a Fréchet space is a suitably modified version of the example of Buche (1968).

*Example* — Consider the Fréchet space $E$ of all functions $x(t)$, $t \in R$, having the following properties : (a) $\sup_{t\in(-b_n, b_n)} |x(t)| \leqslant M_x(b_n)$ where $b_n = O(n)$, $b_n > 0$, and $\{b_n\}$ is a strictly monotonically increasing sequence, and $M_x(b_n) \exp(-\alpha n/b_n^2) \to 0$, For each $\alpha > 0$; (b) the family of seminorms $\{P_n(.)\}$ is defined by

$$P_n(x) = \operatorname*{Sup}_{t \in (-b_n, b_n)} | x(t) | , \; n = 1, 2, 3, \ldots;$$

(c) $x$ is infinitely differentiable at each

$$t \in R \quad \text{and} \quad p_n \left\{ \frac{d^m}{dt^m} x(t)) \right\} \leqslant K p_n(x(t))$$

where $K$ is independent of $m$ and $n$. Consider the differential operator

$$D = \frac{d}{dt}; \text{ then } D^m = \frac{d^m}{dt^m}, \; m = 1, 2, \ldots$$

Also it can be easily checked that

$$T_1(t) = e^{-t} \sum_{k=0}^{\infty} (tD)^k/k! \text{ is a semigroup of operators on } E, \text{ with infinitesimal}$$

generator $A_1 = D - I$.

Exactly in a similar manner we can define another semigroup of operators

$$T_2(t) = e^{-t} \sum_{k=0}^{\infty} (-1)^k (tD)^k/k!$$

With infinitesimal generator $A_2 = -D - I$.

Define $S(t)$ for $\alpha \neq -1$ as

$$S(t) = \frac{\alpha}{1+\alpha} T_1(t) + \frac{1}{1+\alpha} T_2(t).$$

Then $S(t)$ is a family of mixed semigroup of operators on $E$ with its first infinitesimal generator given by $A = (\alpha/1+\alpha)(D-I) + (1/1+\alpha)(-D-I)$.

*Case $\alpha = -1$ and Some Concluding Remarks*

For the case $\alpha = -1$, the representation of the mixed semigroups looks somewhat different from that of Theorem 2.3 and we get the relations which formally take the following shape :

(1) $\quad S^{(1)}(t) = A_0 S(t) + (A - A_0) T(t)$

(2) $\quad S(t) = T(t) + t(A - A_0) T(t)$.

In fact it can be shown that when the underlying space $E$ is a barreled, quasi-complete, locally convex Hausdorff linear topological space then with some conditions as in Theorem 4.1, both the sides of the above relations exist as continuous linear operators from $(s_0, \infty)$ to $L_C(E, E)$ for some $s_0 > 0$. The proofs of the above results are omitted as they follow the method given earlier.

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