Indian J. pure appl. Math., **10**(7) : 867-877, July 1979

ENTROPY AND INFORMATION OF COMPARISON UNDER GENERALIZED ADDITIVITY AND NON-ADDITIVITY

RAMINDER SINGH SONI

Department of Mathematics, S.G.T.B. Khalsa (Evening) College, Dev Nagar, New Delhi 110005

(*Received* 27 *November* 1978)

The well-known additivity of entropy namely

$$H(P^*Q) = H(P) + H(Q)$$

has been generalized in the form

$$H(P^*Q) = \sum_{j=1}^{n} q_j^{\beta} F(P) + \sum_{i=1}^{m} p_i^{\beta} G(Q)$$

where $F(P) = \sum_{i=1}^{m} f(p_i)$, $G(Q) = \sum_{j=1}^{n} g(q_j)$.

Under this generalized additivity measures of entropy have been studied. Non-additive measures are also studied in terms of the solutions

$$H(P^*Q) = \sum_{j=1}^{n} q_j^{\beta} F(P) + \sum_{i=1}^{m} p_i^{\beta} G(Q) + KF(P)\, G(Q).$$

The study has been extended to determine information theoretic measures for comparing two distributions attached with a discrete random variate; which satisfy generalized additivity or generalized non-additivity.

1. INTRODUCTION

Given a discrete finite probability distribution $P = (p_1, p_2, ..., p_m)$, $\sum_{i=1}^{m} p_i = 1$, Shannon's entropy (1948), which is all important in many studies, is given by

$$H(P) = -\sum_{i=1}^{m} p_i \log p_i. \qquad ...(1.1)$$

(unless stated otherwise all logarithms will be considered to the base 2).

If there is another probability distribution $Q = (q_1, q_2, ..., q_n)$, $\sum_{j=1}^{n} q_j = 1$ and $P^*Q = (p_1q_1, p_1q_2, ..., p_1q_n, ..., p_mq_1, p_mq_2, ..., p_mq_n)$, then Shannon's entropy satisfies the additivity

$$H(P^*Q) = H(P) + H(Q) \qquad ...(1.2)$$

Now if

$$H(P) = \sum_{i=1}^{m} h(p_i) \qquad \text{...(1.3)}$$

then (1.2) can be put as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(p_i q_j) = \sum_{i=1}^{m} h(p_i) + \sum_{j=1}^{n} h(q_j). \qquad \text{...(1.4)}$$

The continuous solution of (1.4) obtained by Chaundy and McLeod (1960), with the help of property (1.3) when $h(\frac{1}{2}) = 1$ characterize Shannon's entropy.

Sharma and Taneja (1977a) have generalized (1.4) as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(p_i q_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} q_j^{\beta} h(p_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} p_i^{\beta} h(q_j), \quad \beta > 0 \qquad \text{...(1.5)}$$

and in terms of its continuous solution

$$h(p) = C p^{\beta} \log_a p, \quad a > 1 \qquad \text{...(1.6)}$$

when $h(\frac{1}{2}) = 1$, (1.3) gives a new measure of entropy given by

$$H_{\beta}(P) = -2^{\beta-1} \sum_{i=1}^{m} p_i^{\beta} \log p_i. \qquad \text{...(1.7)}$$

Sharma and Taneja (1977b) have further extended above study by considering in place of (1.5) the additivity governed by the functional equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(p_i q_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} g(q_j)\, h(p_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} g(p_i)\, h(q_j). \qquad \text{...(1.8)}$$

However in (1.5) and (1.8) on the right-hand side same function $h(.)$ occurs as is involved in the definition of entropy.

We shall address ourselves to the generalization

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(p_i q_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} q_j^{\beta} f(p_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} p_i^{\beta} g(q_j), \quad \beta > 0 \qquad \text{...(1.9)}$$

so that if $\sum_{i=1}^{m} f(p_i) = F(P)$ and $\sum_{j=1}^{n} g(q_j) = G(Q)$, the additivity (1.2) is expressed in the generalized form

$$H(P^*Q) = \left(\sum_{j=1}^{n} q_j^{\beta}\right) F(P) + \left(\sum_{i=1}^{m} p_i^{\beta}\right) G(Q). \qquad \text{...(1.10)}$$

We shall go one step still further in considering the functional equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(p_i q_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} q_j^{\beta} f(p_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} p_i^{\beta} g(q_j)$$

$$+ K \sum_{i=1}^{m} \sum_{j=1}^{n} f(p_i) g(q_j), \quad K \neq 0 \qquad \ldots(1.11)$$

which leads to non-additive measure associated with a probability distribution as generalizations of Harvada and Charvat (1967) and Daroczy (1970) entropy of type β.

In section 4 we take up the study of information theoretic measures for comparison of two distributions $P = (p_1, p_2, \ldots, p_m)$ and $P' = (p_1, p_2, \ldots, p_n)$ associated with the same random variate by considering the additivity and non-additivity for such measures leading to modifications of (1.9) and (1.11) in functions of two variables. It may be remarked here that two such well-known measures for comparison are Kullback's (1959) 'relative information' and Kerridge's (1961) 'inaccuracy', which satisfy additivity of the form given in (1.2). Our study is quite rewarding and contains some of the earlier studies as special cases.

2. Solutions of Functional Equations under generalized additivity and non-additivity

We first obtain the most general continuous solutions of functional eqns. (1.9) and (1.11). This is done in the next two theorems.

Theorem 1 — All the continuous solutions of the functional eqn. (1.9) for $x \in [0, 1]$ are given by

$$f(x) = Ax^{\beta} - Cx^{\beta} \log x \qquad \ldots(2.1)$$

$$g(x) = (B - A)\, x^{\beta} - Cx^{\beta} \log x \qquad \ldots(2.2)$$

$$h(x) = Bx^{\beta} - Cx^{\beta} \log x \qquad \ldots(2.3)$$

where A, B, C are arbitrary constants.

Proof : We define the continuous functions F, G, H as

$$F(x) = x^{\beta} f\left(\frac{1}{x}\right), \text{ for } \beta > 0 \text{ etc. and for all real } x \geqslant 1. \qquad \ldots(2.4)$$

Let m, n be any integers greater than or equal to unity. Setting

$$x_i = 1/m \; (i = 1, 2, \ldots, m), \; y_j = 1/n \; (j = 1, 2, \ldots, n)$$

in (1.9) and changing in terms of functions F, G, H, we have

$$H(mn) = F(m) + G(n), \text{ for } m, n \geqslant 1. \qquad \ldots(2.5)$$

For $m = 1$ and then for $n = 1$, (2.5) gives

$$H(n) = F(1) + G(n), \text{ for } n \geqslant 1 \qquad ...(2.6)$$

and

$$H(m) = F(m) + G(1), \text{ for } m \geqslant 1. \qquad ...(2.7)$$

Putting in (2.5) values of $H(m)$ and $G(n)$ in terms of F, we have

$$F(mn) = F(m) + F(n) - F(1), \text{ for integers } m, n \geqslant 1$$

or

$$l(mn) = l(m) + l(n), \text{ for integers } m, n \geqslant 1 \qquad ...(2.8)$$

where $l(x) = F(x) - F(1)$ for real x,

which is Cauchy's functional equation in l for integers. We would extend it for reals.

So consider a rational number $x = m/n$ $(m < n)$ and put

$$\left.\begin{array}{l} x_1 = m/n,\ x_2 = \ldots = x_{n-m+1} = 1/n \\ y_1 = y_2 = \ldots = y_m = 1/m \end{array}\right\} \qquad ...(2.9)$$

in (1.9), we thus obtain (taking $m = n - m + 1$ and n as m)

$$H(n) - G(m) = F(n/m) \qquad ...(2.10)$$

or

$$F(n/m) = F(n) - F(m) + F(1), \text{ for } m < n$$

or

$$l(n/m) = l(n) - l(m), \text{ for } m < n. \qquad ...(2.11)$$

Now (2.8) and (2.11), yield

$$l(xy) = l(x) + l(y), \text{ for all rationals } x, y \geqslant 1. \qquad ...(2.12)$$

Since l is continuous, (2.12) holds for real numbers $x, y \geqslant 1$ and its most general continuous solution is then (Aczel 1966) given by

$$l(x) = C \log x, \text{ for all real } x \geqslant 1, \qquad ...(2.13)$$

where C is an arbitrary constant.

It would be noted that the solution $l(x) = 0$ is included in (2.13) for $C = 0$, so that

$$f(x) = Ax^{\beta} - Cx^{\beta} \log x, \text{ for all real } x \in [0, 1], \beta > 0 \qquad ...(2.14)$$

(assuming by definition that $0 \log 0 = 0$).

Similarly,

$$g(x) = (B - A)\, x^{\beta} - Cx^{\beta} \log x, \text{ for all real } x \in [0, 1], \beta > 0. \quad ...(2.15)$$

Again taking $m = n - m + 1$ and n as m and setting

$$\left.\begin{aligned} x_1 &= m/n,\ x_2 = x_3 = ... = x_{n-m+1} = 1/n \\ y_1 &= 1,\ y_2 = y_3 = ... = y_m = 0 \end{aligned}\right\} \quad ...(2.16)$$

(1.9) gives

$$h(m/n) + (n - m)\, h(1/n) = f(m/n) + (n - m)\, f(1/n) + \left[(m/n)^{\beta} + \frac{(n - m)}{n^{\beta}}\right] g(1). \quad ...(2.17)$$

It follows by taking $x_1 = 1$, $y_1 = 1$, $x_2 = y_2 = 0$ in (1.9) that $h(0) = 0$, since $f(0) = g(0) = 0$.

This in view of (2.4) and (2.5) gives

$$h(x) = f(x) + x^{\beta} g(1) = Bx^{\beta} - Cx^{\beta} \log x \quad ...(2.18)$$

for all rational $x \in [0, 1]$, $\beta > 0$. Using continuity finally to extend result from rationals to reals, we find that (2.18) holds for all real $x \in [0, 1]$.

This completes the proof of the theorem. ∎

Theorem 2 — All the continuous solutions of the functional eqn. (1.11) for $x \in [0, 1]$ are given by

$$h(x) = (M/K)\, x^{\alpha} - x^{\beta}/K \quad ...(2.19)$$

$$f(x) = (N/K)\, x^{\alpha} - x^{\beta}/K \quad ...(2.20)$$

$$g(x) = (M/NK)\, x^{\alpha} - x^{\beta}/K \quad ...(2.21)$$

where M, N, K are arbitrary constants.

PROOF : Using the substitution of preceding theorem in (1.11), we have

$$H(mn) = F(m) + G(n) + KF(m)\, G(n), \text{ for } m, n \geqslant 1. \quad ...(2.22)$$

Substituting values of functions H and G in terms of F, this can be reduced in terms of F alone to give

$$[KF(1) + 1]\, F(mn) = F(m) + F(n) + KF(m)\, F(n) - F(1), \text{ for } m, n \geqslant 1. \quad ...(2.23)$$

Now if $1 + KF(1) = 0$, (2.23) reduces to

$$F(m) + F(n) + KF(m)\, F(n) - F(1) = 0$$

or

$$[1 + KF(m)]\,[1 + KF(n)] = 0 \quad \text{as} \quad K \neq 0.$$

So that

$$F(m) = -\,1/K, \text{ a constant function for } m \geqslant 1. \qquad ...(2.24)$$

However if $1 + KF(1) \neq 0$, take $C = K/(1 + KF(1))\ (\neq 0)$ and set for any real number m,

$$C\,[F(m) - F(1)] + 1 = l(m) \qquad ...(2.25)$$

Equation (2.23) now gives

$$l(mn) = l(m)\,.\,l(n), \text{ for integers } m, n \geqslant 1. \qquad ...(2.26)$$

Extending to rationals as before in (2.9), we get

$$[1 + KF(1)]\,F(n) = F(m) + F(n/m) + KF(m)\,F(n/m) - F(1), \quad \text{for } 0 < m \leqslant n. \qquad ...(2.27)$$

Now using the substitution in (2.25), this can be put as

$$l(n) = l(m)\,.\,l(n/m), \text{ for } 0 < m \leqslant n. \qquad ...(2.28)$$

Now (2.26) and (2.28), yield

$$l(xy) = l(x)\,.\,l(y), \text{ for all rational numbers } x, y \geqslant 1. \qquad ...(2.29)$$

Since, l is continuous, (2.29) holds for real numbers $x, y \geqslant 1$, and it follows (Aczel 1966) that

$$l(x) = x^{\lambda} \quad \text{or} \quad l(x) = 0, \text{ for all real } x \geqslant 1. \qquad ...(2.30)$$

From (2.25) and (2.30), we get

$$F(x) = (N/K)\,x^{\lambda} - 1/K, \text{ for all real } x \geqslant 1. \qquad ...(2.31)$$

This includes the case for $l(x) = 0$ also as can be seen by taking $N = 1 + KF(1) = 0$ and this has been discussed earlier for integers in (2.24).

Thus

$$f(x) = x^{\beta}F(1/x) = (N/K)\,x^{\beta-\lambda} - x^{\beta}/K, \text{ for all real } x \in [0, 1]$$
$$= (N/K)\,x^{\alpha} - x^{\beta}/K, \text{ where } \alpha = \beta - \lambda, \text{ for all real } x \in [0, 1]\ \alpha, \beta > 0. \qquad ...(2.32)$$

Similarly

$$g(x) = (M/NK)\,x^{\alpha} - x^{\beta}/K, \text{ for all real } x \in [0, 1], \alpha, \beta > 0. \qquad ...(2.33)$$

Again under the substitution in (2.16), (1.11) gives

$$h(m/n) + (n - m)\, h(1/n) = f(m/n) + (n - m)\, f(1/n)$$

$$+ g(1)\left[(m/n)^{\beta} + \frac{(n-m)}{n^{\beta}} + Kf(m/n) + K(n-m)\, f\left(\frac{1}{n}\right)\right]$$

since $h(0) = f(0) = g(0) = 0$ ...(2.34)

or

$$h(m/n) = f(m/n)\,[1 + Kg(1)] + g(1)\,(m/n)^{\beta}$$

or

$$h(x) = (M/K)\, x^{\alpha} - x^{\beta}/K, \text{ for any rational number } x \in [0, 1],\ \alpha, \beta > 0. \qquad ...(2.35)$$

Using continuity finally, we find that (2.35) holds for all real $x \in [0, 1]$.

This completes the proof of the theorem. ∎

3. Entropies Satisfying Generalized Additivity and Non-additivity

Let $P = (p_1, p_2, ..., p_m)$, $\sum_{i=1}^{m} p_i = 1$ be a probability distribution associated with a discrete random variate $X = (x_1, x_2, ..., x_m)$. The entropies are sum functions of the distribution (see (1.3)). If $H_A^{\beta}(P)$ and $H_N^{\beta}(P)$ denote respectively the entropies satisfying generalized additivity (1.10) and non-additivity corresponding to (1.11) then using (1.3) in terms of functions determined in Theorems 1 and 2

$$H_A^{\beta}(P) = B \sum_{i=1}^{m} p_i^{\beta} - C \sum_{i=1}^{m} p_i^{\beta} \log p_i, \quad \beta > 0 \qquad ...(3.1)$$

and

$$H_N^{\beta}(P) = (M/K) \sum_{i=1}^{m} p_i^{\alpha} - (1/K) \sum_{i=1}^{m} p_i^{\beta}, \quad \alpha, \beta > 0 \qquad ...(3.2)$$

where base of the logarithm is arbitrary (> 1), and B, C, M, K are arbitrary constants. These constants may be determined by requiring these measures to satisfy some reasonable boundary and normalizing conditions.

Theorem 3 — The measures (3.1) and (3.2) associated with a probability distribution P of a discrete random variate under the conditions

$$H(1, 0) = 0, \ H(\tfrac{1}{2}, \tfrac{1}{2}) = 1 \qquad ...(3.3)$$

are the following entropies

$$H_A^{\beta}(P) = -(1/2^{1-\beta}) \sum_{i=1}^{m} p_i^{\beta} \log_2 p_i \qquad ...(3.4)$$

and

$$H_N^{\beta}(P) = \left[\sum_{i=1}^{m} p_i^{\alpha} - \sum_{i=1}^{m} p_i^{\beta}\right][2^{1-\alpha} - 2^{1-\beta}]^{-1}. \qquad \ldots(3.5)$$

Note : (3.4) and (3.5) are two solutions of the functional eqn. (1.8).

PROOF : (3.1) and (3.2) with the help of (3.3) give $B = 0$,

$$C = -\frac{1}{2^{1-\beta}\log 2}, \quad M = 1 \quad \text{and} \quad 0 \neq K = (2^{1-\alpha} - 2^{1-\beta}).$$

Substituting the values of B, C and M, K in (3.1) and (3.2) respectively, we get (3.4) and (3.5). ∎

Particular Cases

(1) If we take $\alpha = 1$ in (3.5), we get

$$H_N^{(1,\beta)}(P) = \left[\sum_{i=1}^{m} p_i^{\beta} - 1\right](2^{1-\beta} - 1)^{-1} \qquad \ldots(3.6)$$

which has been studied by Daroczy (1970), Harvada and Charvat (1967).

(2) If we take $\alpha = 2\beta - 1$ in (3.5), we get

$$H_N^{(2\beta-1,\beta)}(P) = \left[\sum_{i=1}^{m} p_i^{2\beta-1} - \sum_{i=1}^{m} p_i^{\beta}\right](2^{2-2\beta} - 2^{1-\beta})^{-1}. \qquad \ldots(3.7)$$

which has been obtained differently by Sharma and Taneja (1977a).

4. MEASURES OF COMPARISON UNDER GENERALIZED ADDITIVITY AND NON-ADDITIVITY

If there are two distributions $P = (p_1, p_2, \ldots, p_m)$,

$$\sum_{i=1}^{m} p_i = 1 \text{ and } Q = (q_1, q_2, \ldots, q_m), \sum_{i=1}^{m} q_i = 1$$

associated with a discrete random variate then a measure of their comparison may be denoted by $H(P/Q)$, such that it satisfies the sum-property

$$H(P/Q) = \sum_{i=1}^{m} h(p_i, q_i) \qquad \ldots(4.1)$$

where $h(p_i, q_i)$ is a function to be determined under the generalized additivity

$$H(P_1^* P_2/Q_1^* Q_2) = \left(\sum_{j=1}^{n} p_{2j}^{\beta}\right) F(P_1/Q_1) + \left(\sum_{i=1}^{m} p_{1i}^{\beta}\right) G(P_2/Q_2) \qquad \ldots(4.2)$$

where $F(P_1/Q_1) = \sum_{i=1}^{m} f(p_{1i}, q_{1i})$ and $G(P_2/Q_2) = \sum_{j=1}^{n} g(p_{2j}, q_{2j})$

For characterizing measures of comparison of type β associated with two probability distributions, we shall study the functional equations

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(x_i y_j, u_i v_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_j^{\beta} f(x_i, u_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^{\beta} g(y_j, v_j) \qquad \ldots(4.3)$$

and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} h(x_i y_j, u_i v_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_j^{\beta} f(x_i, u_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^{\beta} g(y_j, v_j) + K \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, u_i)\, g(y_j, v_j) \qquad \ldots(4.4)$$

where $\quad x_i, y_j, u_i, v_j \geqslant 0, \ \sum_{i=1}^{m} x_i = 1 = \sum_{j=1}^{n} y_j, \ \sum_{i=1}^{m} u_i \leqslant 1$ and $\sum_{j=1}^{n} v_j \leqslant 1.$

For obtaining solutions of functional eqns. (4.3) and (4.4), we may use the substitutions

$$H(x, y) = x^{\beta} h(1/x, 1/y) \text{ etc. for } \beta > 0.$$

Rest of the procedure is also similar to one followed in Theorems 1 and 2 (refer also Sharma and Soni 1975). We give without proofs, the result in the next theorem.

Theorem 4 — All the continuous solutions of the functional eqn. (4.3) for $x, y \in [0, 1]$ are given by

$$h(x, y) = Bx^{\beta} - Cx^{\beta} \log x + Dx^{\beta} \log (x/y), \quad \beta > 0 \qquad \ldots(4.5)$$

$$f(x, y) = (B - A)\, x^{\beta} - Cx^{\beta} \log x + Dx^{\beta} \log (x/y), \quad \beta > 0 \qquad \ldots(4.6)$$

and

$$g(x, y) = Ax^{\beta} - Cx^{\beta} \log x + Dx^{\beta} \log (x/y), \quad \beta > 0 \qquad \ldots(4.7)$$

where A, B, C, and D are arbitrary constants.

Theorem 5 — All the continuous solutions of the functional eqn. (4.4) for $x, y \in [0, 1]$ are given by

$$h(x, y) = (M/K)\, x^{\beta-\lambda-\mu} y^{\mu} - x^{\beta}/K \qquad \ldots(4.8)$$

$$f(x, y) = (N/K)\, x^{\beta-\lambda-\mu} y^{\mu} - x^{\beta}/K \qquad \ldots(4.9)$$

and

$$g(x, y) = (M/NK)\, x^{\beta-\lambda-\mu} y^{\mu} - x^{\beta}/K \qquad \ldots(4.10)$$

$$\left.\right\} \beta > 0, \ \mu > 0, \ \beta - \lambda - \mu > 0$$

where M, N, K are arbitrary constants.

Let $H_A^{\beta}(P/Q)$ and $H_N^{\beta}(P/Q)$ denote respectively the measures of comparison satisfying generalized additivity (4.2) and non-additivity corresponding to (4.4), then using (4.1) in terms of functions determined in the above theorems,

$$H_A^{\beta}(P/Q) = B \sum_{i=1}^{m} p_i^{\beta} - C \sum_{i=1}^{m} p_i^{\beta} \log p_i + D \sum_{i=1}^{m} p_i^{\beta} \log (p_i/q_i),\ \beta > 0 \qquad ...(4.11)$$

and

$$H_N^{\beta}(P/Q) = (M/K) \sum_{i=1}^{m} p_i^{\beta-\lambda-\mu} q_i^{\mu} - (1/K) \sum_{i=1}^{m} p_i^{\beta},\ \beta-\lambda-\mu > 0,\ \beta > 0 \qquad ...(4.12)$$

where base of the logarithm is arbitrary (> 1), and B, C, M, K are arbitrary constants. These constants may by determined by requiring these measures to satisfy some reasonable boundary and normalizing conditions.

Theorem 6 — The measure (4.11) associated with two probability distributions P and Q of a discrete random variate under the conditions

$$H(\{1, 0\} \,\|\, \{1, 0\}) = 0,\ H(\{1, 0\} \,\|\, \{\tfrac{1}{2}, \tfrac{1}{2}\}) = 1 \qquad ...(4.13)$$

and

$$H(\{\tfrac{1}{2}, \tfrac{1}{2}\} \,\|\, \{\tfrac{1}{2}, \tfrac{1}{2}\}) = 0 \qquad ...(4.14)$$

is the Kullback's information of type β, given by

$$H_A^{\beta}(P/Q) = \frac{1}{2^{1-\beta}} \sum_{i=1}^{m} p_i^{\beta} \log 2\,(p_i/q_i). \qquad ...(4.15)$$

PROOF : (4.11) with the help of (4.13) and (4.14) gives

$$B = 0,\ C = 0 \ \text{ and } \ D = \frac{1}{2^{1-\beta} \log 2}.$$

Substituting the values of these constants in (4.11), we get (4.15). ∎

Theorem 7 — The measure (4.11) associated with two probability distributions P and Q of a discrete random variate under the conditions (4.13) and

$$H(\{\tfrac{1}{2}, \tfrac{1}{2}\} \,\|\, \{\tfrac{1}{2}, \tfrac{1}{2}\}) = 1 \qquad ...(4.16)$$

is the Kerridge's inaccuracy of type β, given by

$$H_A^{P}(P/Q) = -\frac{1}{2^{1-\beta}} \sum_{i=1}^{m} p_i^{\beta} \log q_i. \qquad ...(4.17)$$

PROOF : (4.11) with the help of (4.13) and (4.16) gives

$$B = 0,\ C = -\frac{1}{2^{1-\beta}\log 2} \text{ and } D = \frac{1}{2^{1-\beta}\log 2}.$$

Substituting the values of these constants in (4.11), we get (4.17). ∎

Theorem 8 — The measure (4.12) associated with two distributions P and Q of a discrete random variate under the condition (4.13), is given by

$$H_N^{\beta}(P/Q) = \left[\sum_{i=1}^{m} p_i^{\beta-\lambda-\mu} q_i^{\mu} - \sum_{i=1}^{m} p_i^{\beta}\right](2^{1-\beta-\mu} - 2^{1-\beta})^{-1}. \quad \ldots(4.18)$$

PROOF : (4.12) with the help of (4.13) gives $M = 1$ and

$0 \neq K = 2^{1-\beta-\mu} - 2^{1-\beta}$.

Substituting the values of these constants in (4.12), we get (4.18). ∎

Particular Case

If we take $\lambda = 1 - \alpha$ and $\mu = \alpha - \beta$ in (4.18), we get

$$H_N^{\beta}(P/Q) = \left[\sum_{i=1}^{m} p_i^{2\beta-1} q_i^{\alpha-\beta} - \sum_{i=1}^{m} p_i^{\beta}\right](2^{\beta-\alpha} - 1)^{-1} \cdot 2^{1-\beta} \quad \ldots(4.19)$$

which is a two parametric family of inaccuracy measures and be studied independently also.

Some properties of these measures can be studied in a straightforward manner which we are not including to save space.

ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to Dr Bhu Dev Sharma, Department of Mathematics, University of Delhi, Delhi, for guidance and encouragement in carrying out this research work and to S. Surjeet Singh Randhawa Vice-Principal, S.G.T.B. Khalsa (Evening) College, New Delhi, for encouragement.

REFERENCES

Aczel, J. (1966). Lectures on Functional Equations and their Applications. Academic Press, New York.

Chaundy, T. W., and McLeod, J. B. (1960). On a Functional Equation. *Proc. Edinb. math. Soc., Edinb. Math. Notes*, 43, 7-8.

Darcozy, Z. (1970). Generalized information functions. *Inform. Control*, **16**, 36-51.

Harvada, J., and Charvat, F. (1967). Quantification method of classification processes, The concept of structural α-entropy. *Kybernetika*, 3, 30-35.

Kerridge, D. F. (1961). Inaccuracy and inference. *J. R. Stat. Soc.*, Series B, **23**, 184-94.

Kullback, S. (1959). Information Theory and Statistics. John Wiley and Sons, Inc., New York.

Sharma, B. D., and Soni, R. S. (1975). Additive and non-additive measure of information of comparison and information improvement. *EIK (Germany)*, **11**, 489-508.

Sharma, B. D., and Taneja, I. (1977a). A generalized functional equation in information theory. (Preprint).

———— (1977b). Three new measures of entropy. *E.I.K. (Germany)*, **13**, 271-85.

Shannon, C. E. (1948). A mathematical theory of communications. *Bell System Tech. J.*, **27**, 378-426 and 623-56.