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A NEW GENERALIZATION OF TRICOMI'S THEOREM

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A theorem involving Whittaker transform and Meijer-Laplace transform is presented in this paper. Various particular cases of this theorem together with some examples illustrating its utility are also discussed. The theorem is similar to Tricomi's theorem involving Laplace transforms.

§1. A generalization of the classical Laplace transform

$$F(s) = s \int_0^\infty e^{-st} f(t)\, dt \qquad \text{...(1.1)}$$

has been given by Varma (1947) in the form

$$F(s) = s \int_0^\infty (2st)^{-1/4}\, W_{k,\mu}(2st)\, f(t)\, dt \qquad \text{...(1.2)}$$

known as Whittaker transform and represented symbolically by $F(s;\, k,\, \mu) \underset{\mu}{\overset{k}{=}} f(t)$ or briefly by

$$F(s) \underset{\mu}{\overset{k}{=}} f(t).$$

The Whittaker transform (1.2) has the following particular cases (Varma 1947) :

(i) Taking $k = 0$ we get the K_m-transform, (ii) taking $k = \frac{1}{2} + l/2 + n$ and $\mu = \pm\, l/2$, n being a positive integer, we get L_n^l-transform, (iii) taking $k = n/2 + \frac{1}{4}$ and $\mu = \pm\, \frac{1}{4}$ we get the D_n-transform, and (iv) replacing in the K_m-transform $f(t)$ by $t^{1/4} f(t)$ and $F(s)$ by $(2s)^{-1/4} F(s)$ we get another generalization of Laplace transform due to Meijer (1941b). When $k = \frac{1}{4}$ and $\mu = \pm\, \frac{1}{4}$, (1.2) reduces to (1.1).

Another generalization of (1.1) has been given by Bhise (1959) in the form

$$F(s) = s \int_0^\infty G_{m,m+1}^{m+1,0}\left(st \,\middle|\, \begin{matrix}\{\alpha_m + \eta_m\} \\ \{\eta_m\},\, \rho\end{matrix}\right) f(t)\, dt \qquad \text{...(1.3)}$$

known as Meijer-Laplace transform, where the symbol $\{A_m\}$ represents the set of parameters $A_1, \ldots, A_m$ and (1.3) is symbolically denoted by

$$F(s) = G\,[f(t);\, \alpha_m, \eta_m, \rho].$$

The Meijer's G-function in (1.3) is defined as

$$G_{p,q}^{m,n}\left(z \,\Big|\, \begin{matrix}\{a_p\}\\ \{b_q\}\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\xi)}\, z^{\xi}\, d\xi$$

where z may be real or complex but not identically zero and an empty product is to be interpreted as unity. The quantities m, n, p, q are positive integers or zero satisfying the conditions $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; a_j $(j = 1, \ldots, p)$ and b_j $(j = 1, \ldots, q)$ are complex numbers such that no pole of $\Gamma(b_j - \xi)$ $(j = 1, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_j + \xi)$ $(j = 1, \ldots, n)$ and the poles are simple. The contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ and has the poles of $\Gamma(1 - a_j + \xi)$, $(j = 1, \ldots, n)$, to its left and the poles of $\Gamma(b_j - \xi)$, $(j = 1, \ldots, m)$, to its right.

With $\alpha_i = 0$ $(i = 1, \ldots, m-1)$, $\rho = 0$, $\alpha_m = \frac{1}{2} - \mu - k$ and $\eta_m = 2\mu$, (1.3) reduces to Varma transform

$$F(s) = s\int_0^\infty (st)^{\mu-(1/2)} e^{-(st/2)} W_{k,\mu}(st)\, f(t)\, dt \qquad \ldots(1.4)$$

which is another generalization of (1.1) by Varma (1951) and symbolically denoted by

$$F(s) = W\,[f(t);\, k, \mu].$$

With $\alpha_i = 0$ $(i = 1, \ldots, m-1)$, $\alpha_m = \rho = -\mu - k$, $\eta_m = \mu - k$, (1.3) reduces to Meijer (1941a) transform

$$F(s) = s\int_0^\infty (st)^{-k-(1/2)} e^{-st/2} W_{k+(1/2),\mu}(st)\, f(t)\, dt \qquad \ldots(1.5)$$

symbolically denoted by $f(t) \xrightarrow[\mu]{k+\frac{1}{2}} F(s)$.

While, with $\alpha_i = 0$ $(i = 1, \ldots, m)$ and $\rho = 0$, (1.3) reduces to (1.1).

In this paper we establish a theorem which is a generalization of the well-known Tricomi's theorem, involving Whittaker transform (1.2) and Meijer-Laplace transform (1.3). Various particular cases of this theorem together with some examples illustrating its utility are discussed. Some of the results derived in this paper are believed to be new.

§2. In this section we first establish a result to be used in section 3 to establish the main theorem of this paper. Let $F(s)$ be the Meijer-Laplace transform as given by (1.3) of a function $f(t)$ chosen as

$$f(t) = t^c G^{lm+l,nm}_{nm+lm,\,lm+l+nm+n}\left(\frac{b^l t^n}{l^l n^n}\middle|\begin{matrix}\{\Delta(n, -\alpha_m-\eta_m-c)\},\{\Delta(l, a_m+c_m)\}\\ \{\Delta(l, c_m)\},\Delta(l,d),\{\Delta(n,-\eta_m-c)\},\Delta(n,-c-\rho)\end{matrix}\right). \quad ...(2.1)$$

Then

$$s^{-c} G^{m+1,0}_{m,m+1}\left(bs^{-n/l}\middle|\begin{matrix}\{a_m+c_m\}\\ \{c_m\}, d\end{matrix}\right)$$

$$= (2\pi)^{-(l-n)/2}\, n^{-\rho+\sum_1^m \alpha_i - c - \frac{1}{2}}\, l^{d-\sum_1^m a_i+\frac{1}{2}} F(s) \quad ...(2.2)$$

provided

$\operatorname{Re}[c+1+\min(\eta_j, \rho)+n\min(c_i/l, d/l)] > 0$ for $j, i = 1, ..., m$;

$\operatorname{Re}[c+1+n(-\alpha_h-\eta_h)] > n+1$ for $h = 1, ..., m$;

$|\arg s| < \pi/2$; $l > n$; $|\arg\{b^l/(l^l n^n)\}| < (l-n)\pi/2$ and

$\operatorname{Re}[\sum_1^m \alpha_h - \rho] + 1 + n[\operatorname{Re}(-\alpha_i-\eta_i-c+j-1)/n - c] > 0$

for $i = 1, ..., m$; $j = 1, ..., n$.

Here we state only the brief outline of the proof of (2.2). First we substitute the value of $f(t)$ given by (2.1) in (1.3) using the formula given by Saxena (1960). We have

$$F(s) = (2\pi)^{(l-n)/2}\, n^{\rho-\sum_1^m \alpha_i+c+\frac{1}{2}}$$

$$\times\, G^{l(m+1),0}_{lm,l(m+1)}\left(\frac{b^l}{l^l s^n}\middle|\begin{matrix}\{\Delta(l, a_m+c_m)\}\\ \{\Delta(l, c_m)\}, \Delta(l,d)\end{matrix}\right) s^{-c} \quad ...(2.3)$$

where the symbol of the type $\{\Delta(l, a_m+c_m)\}$ represents the set of parameters $\Delta(l, a_1+c_1), ..., \Delta(l, a_m+c_m)$. Now using the multiplication formula for G-function given by Saxena (1960), namely

$$G^{lm,ln}_{lp,lq}\left[\{zl^{-(q-p)}\}^l\middle|\begin{matrix}\{\Delta(l, a_p)\}\\ \{\Delta(l, b_q)\}\end{matrix}\right]$$

$$= (2\pi)^{(l-1)(m+m-\frac{1}{2}p-\frac{1}{2}q)}\, l^{-\sum_1^q b_j+\sum_1^p a_j-\frac{1}{2}p+\frac{1}{2}q-1}\, G^{m,n}_{p,q}\left[z\middle|\begin{matrix}\{a_p\}\\ \{b_q\}\end{matrix}\right] \quad ...(2.4)$$

and simplifying further, (2.2) can easily be established.

§3. We now state and prove the main theorem.

Theorem — If $F(s) \overset{k}{\underset{\mu}{=}} f(y)$, then

$$s^{(n/l)-c} F(s^{-n/l}) = (2\pi)^{-\frac{1}{2}(l-n)}\, n^{-\rho+\sum_1^m \alpha_i - c - \frac{1}{2}}$$

$$\times\, l^{1/2} G\left[t^c \int_0^\infty K\left(\frac{y^l t^n}{l^l n^n}\right) f(y)\, dy;\ \alpha_m, \eta_m, \rho \right] \qquad \ldots(3.1)$$

where $$K(x) = \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+\mu-k)_r\,(2l)^{(4\mu+1+4r)/4}}{(2\mu+1)_r\, r!}$$

$$\times\, G^{l,nm}_{nm,l+nm+n}\left(x \,\middle|\, \begin{matrix} \{\triangle(n, -\alpha_m-\eta_m-c)\} \\ \triangle(l, \mu+\frac{1}{4}+r), \{\triangle(n, -c-\eta_m)\}, \triangle(n, -c-\rho) \end{matrix}\right) \qquad \ldots(3.2)$$

provided $|\arg s| < \pi/2$, $l > n$, $\mathrm{Re}\,(s) > 0$, (2μ) is not zero or an integer,

$\mathrm{Re}\,[c+1+\min(\eta_j, \rho)] > 0,\ j = 1, \ldots, m;$

$\mathrm{Re}\,[c+1+n(-\alpha_h-\eta_h)] > n+1,\ h = 1, \ldots, m;$

$\mathrm{Re}\,[\sum_1^m \alpha_j - \rho] + 1 + n\,[\mathrm{Re}\,((-\alpha_h-\eta_h-c)/n) - c] > 0,\ h = 1, \ldots, m$

and the integrals involved are absolutely and uniformly convergent.

PROOF : We have

$$F(s) \overset{k}{\underset{\mu}{=}} f(y) = s \int_0^\infty (2sy)^{-1/4} W_{k,\mu}(2sy)\, f(y)\, dy.$$

Replacing s by $s^{-n/l}$ and multiplying by $s^{(n/l)-c}$ we have

$$s^{(n/l)-c}F(s^{-n/l}) = \int_0^\infty (2y)^{-1/4} s^{(n/4l)-c} W_{k,\mu}(2s^{-n/l}y)\, f(y)\, dy$$

$$= \int_0^\infty \Bigg[\sum_{\mu,-\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+\mu-k)_r\,(2y)^{(4\mu+1+4r)/4}}{(2\mu+1)_r\, r!}$$

$$\times\, s^{-\,c+n(4\mu+1+4r)/4l]} \exp(-ys^{-n/l}) \Bigg] f(y)\, dy \qquad \ldots(3.3)$$

by expressing the Whittaker function inside the integral in its series form.

Now, if we put in (2.2) $a_i = 0$ ($i = 1, \ldots, m$) and $d = 0$ and simplify we get

$$s^{-c} \exp(-bs^{-n/l}) = (2\pi)^{-(l-n)/2} n^{-\rho+\sum_1^m \alpha_i - c - \frac{1}{2}} l^{\frac{1}{2}} G\Big[t^c G^{l,nm}_{nm,l+nm+n} \left(\frac{b^l t^n}{l^l n^n} \middle| \begin{matrix} \{\Delta(n, -\alpha_m - \eta_m - c)\} \\ \Delta(l, 0), \{\Delta(n, -\eta_m - c)\}, \Delta(n, -c-\rho) \end{matrix}\right); \alpha_m, \eta_m, \rho\Big] \quad \ldots(3.4)$$

provided $|\arg s| < \pi/2$, $l > n$, $\mathrm{Re}(s) > 0$,

$\mathrm{Re}[c + 1 + \min(\eta_j, \rho)] > 0$, $j = 1, \ldots, m$;

$\mathrm{Re}[c + 1 + n(-\alpha_h - \eta_h)] > n + 1$, $h = 1, \ldots, m$ and

$\mathrm{Re}[\sum_1^m \alpha_j - \rho] + 1 + n[\mathrm{Re}(-\alpha_h - \eta_h - c)/n - c] > 0$, $h = 1, \ldots, m$.

Hence, replacing c by $c + n(\mu + \frac{1}{4} + r)/l$ and b by y in (3.4) and substituting in (3.3), interchanging the order of integration we get (3.1) after some simplification and interpreting the integral on the r.h.s. of (3.3) in Meijer-Laplace transform by (3.4).

We know that if the series and the integrals are absolutely and uniformly convergent the term by term interpretation is justified. The absolute and uniform convergence of the series in (3.2) can be established with the help of the inequality

$$G^{l,nm}_{nm,l+nm+n}\left(x \middle| \begin{matrix} \{\Delta(n, -\alpha_m - \eta_m - c)\} \\ \Delta(l, \mu + \frac{1}{4} + r), \{\Delta(n, -c - \eta_m)\}, \Delta(n, -c-\rho) \end{matrix}\right)$$

$$< x^{(4\mu+1+4r)/4} G^{l,0}_{0,l}\left(x \middle| \begin{matrix} - \\ \Delta(l, 1) \end{matrix}\right)$$

which can be seen to be true after expanding both the G-functions in series of hypergeometric functions, expressing the hypergeometric functions in their series form and comparing the two results term by term, after taking into account that

(i) $$\frac{\prod_{i=1}^{m}\prod_{j=1}^{n} \Gamma(A_{ij} + r/l)}{\prod_{i=1}^{m}\prod_{j=1}^{n} \Gamma(B_{ij} + r/l) \prod_{j=1}^{n} \Gamma(C_j + r/l)} < 1,$$

(ii) $$\frac{\prod_{i=1}^{m}\prod_{j=1}^{n} (A_{ij} + r/l)_k}{\prod_{i=1}^{m}\prod_{j=1}^{n} (B_{ij} + r/l)_k \prod_{j=1}^{n} (C_j + r/l)_k} < 1,$$

(iii) the power of x on the left-hand side is greater than that on the right-hand side and

(iv) $|x| < 1$ so that each term on the left-hand side is less than the corresponding term on the right-hand side.

The absolute and uniform convergence of the integrals has been assumed. It may be noted further that $K(y^l t^n/l^l n^n)$ is a continuous function of t and y for t, $y > 0$.

We may state here that in view of Section 1 several interesting generalizations of Tricomi's theorem can be obtained by giving suitable values to the parameters.

§4. In this section we give two examples to illustrate the utility of the theorem.

Example (i) — Let $f(y) = y^a$.

We first require the value of the integral

$$\int_0^\infty y^a G^{l,nm}_{nm;l+nm+n} \left(\frac{t^n}{l^l n^n} y^l \,\middle|\, \begin{matrix} \{\triangle(n, -\alpha_m - \eta_m - c)\} \\ \triangle(l, \mu + \tfrac{1}{4} + r), \{\triangle(n, -\eta_m - c)\}, \triangle(n, -c-\rho) \end{matrix} \right) dy$$

which can be seen by the substitution $y^l = z$ and using (3.2.1) of Mathai and Saxena (1973 p. 79) to be equal to

$$\frac{1}{l} (l^l n^n/t^n)^{(a+1)/l} \chi\left(\frac{a+1}{l}\right) \qquad \ldots(4.1)$$

where

$$\chi(\xi) = \frac{\prod_{j=1}^{l} \Gamma(\mu + \tfrac{1}{4} + r + j - 1)/l + \xi) \prod_{j=1}^{n} \prod_{i=1}^{m} \Gamma(1 - (1/n)(-\alpha_i - \eta_i - c + j - 1) - \xi)}{\prod_{j=1}^{n} \prod_{i=1}^{m} \Gamma(1 - (1/n)(-c - \eta_i + j - 1) - \xi) \prod_{j=1}^{n} \Gamma(1 - (1/n)(-c - \rho + j - 1) - \xi)} \qquad \ldots(4.2)$$

provided $l > n$, $\left|\arg \frac{t^n}{l^l n^n}\right| < \frac{1}{2}(l-n)\pi$,

$$-\operatorname{Re}\left[\frac{1}{l}(\mu + \tfrac{1}{4} + r)\right] < \operatorname{Re}\left(\frac{a+1}{l}\right)$$

$$< 1 - \max \operatorname{Re}\left[\frac{l}{n}(-\alpha_h - \eta_h - c)\right], h = 1, \ldots, m.$$

Now (Bose 1949)

$$F(s) \overset{k}{\underset{\mu}{=}} y^{a} = \frac{\Gamma(a \pm \mu + \frac{5}{4})}{2(2s)^{a}\,\Gamma(a - k + \frac{7}{4})}\ {}_2F_1\left[\begin{matrix} a \pm \mu + \frac{5}{4} \\ a - k + \frac{7}{4} \end{matrix}; \tfrac{1}{2}\right]. \qquad \text{...(4.3)}$$

Therefore, applying (3.1) to $F(s)$ with the help of (4.1) we get

$$G\left[t^{c-(n(a+1)/l)} \sum_{\mu,-\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2} - \mu - k)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2} + \mu - k)_r\,(2l)^{(4\mu+1+4r)/4}}{(2\mu + 1)_r\, r!} \times \chi\left(\frac{a+1}{l}\right);\ \alpha_m, \eta_m, \rho\right]$$

$$= l^{-a-\frac{1}{2}}\, n^{-(n/l)(a+1)+\rho-\sum_{1}^{m}\alpha_i+c+\frac{1}{2}}\,(2\pi)^{+\frac{1}{2}(l-n)} \times \frac{s^{(n/l)(a+1)-c}}{2^{n+1}} \frac{\Gamma(a \pm \mu + \frac{5}{4})}{\Gamma(a - k + \frac{7}{4})}\ {}_2F_1\left[\begin{matrix} a \pm \mu + \frac{5}{4} \\ a - k + \frac{7}{4} \end{matrix}; \tfrac{1}{2}\right] \qquad \text{...(4.4)}$$

under the conditions mentioned with (4.1).

Example (ii) — Let $f(y) = y^{a}e^{-qy}$

Here we first require the value of the integral

$$\int_0^{\infty} y^{a}e^{-qy} G^{l,nm}_{nm,l+nm+n}\left(\frac{t^n}{l^l n^n}\, y^l \,\middle|\, \begin{matrix} \{\triangle(n, -\alpha_m - \eta_m - c)\} \\ \triangle(l, \mu + \frac{1}{4} + r), \{\triangle(n, -\eta_m - c)\}, \triangle(n, -c-\rho) \end{matrix}\right) dy$$

which can be seen, by using the results of Saxena (1960) to be equal to

$$q^{-a-1}(2\pi)^{-(1/2)(l-1)}\, l^{a+(1/2)}\, G^{l,nm+l}_{nm+l,l+nm+n}\left(\frac{t^n}{q^l n^n} \,\middle|\, \begin{matrix} \{\triangle(n, -\alpha_m - \eta_m - c)\}, \triangle(l, -a) \\ \triangle(l, \mu + \frac{1}{4} + r), \{\triangle(n, -\eta_m - c)\}, \triangle(n, -c-\rho) \end{matrix}\right) \qquad \text{...(4.5)}$$

provided $R\,[a + 1 + \mu + \frac{1}{4} + r] > 0$,

$R\left[a + 1 - \frac{l}{n}(\alpha_h + \eta_h + c)\right] > l + 1,\ h = 1, \ldots, m;$

$|\arg q| < \pi/2,\ l > n$ and $\left|\arg \frac{t^n}{l^l n^n}\right| < \frac{1}{2}(l - n)\,\pi.$

Now (Bose 1949)

$$F(s) \overset{k}{\underset{\mu}{=}} y^{a}e^{-qy} = \frac{\Gamma(a \pm \mu + \frac{5}{4})}{2(2s)^{a}\,\Gamma(a - k + \frac{7}{4})}\ {}_2F_1\left[\begin{matrix} a \pm \mu + \frac{5}{4} \\ a - k + \frac{7}{4} \end{matrix}; \tfrac{1}{2} - \frac{q}{2s}\right] \qquad \text{...(4.6)}$$

provided $R(a \pm \mu + \frac{5}{4}) > 0$, $R(s) > 0$ and $|s| > |q|$. Therefore, applying (3.1) to $F(s)$ with the help of (4.5) we get

$$G\Big[\sum_{\mu,-\mu} \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-k)} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}+\mu-k)_r\,(2l)^{(4\mu+1+4r)/4}}{(2\mu+1)_r\,r!}$$

$$t^c G^{l,nm+l}_{nm+l,l+nm+n}\left(\frac{t^n}{n^n q^l}\middle| \begin{matrix}\{\triangle(n,-\alpha_m-\eta_m-c)\},\triangle(l,-a)\\ \triangle(l,\mu+\frac{1}{4}+r),\{\triangle(n,-\eta_m-c)\},\triangle(n,-c-\rho)\end{matrix}\right);\alpha_m,\eta_m,\rho\Big]$$

$$= s^{(n/l)(a+1)-c}\left(\frac{q}{2l}\right)^{a+1}(2\pi)^{l-\frac{1}{2}(n+1)}\,n^{\rho-\sum_1^m \alpha_i+c+\frac{1}{2}}$$

$$\times \frac{\Gamma(a\pm\mu+\frac{5}{4})}{\Gamma(a-k+\frac{7}{4})}\,{}_2F_1\left[\begin{matrix}a\pm\mu+\frac{5}{4}\\ a-k+\frac{7}{4}\end{matrix};\frac{1}{2}-\frac{1}{2}qs^{n/l}\right] \qquad \ldots(4.7)$$

provided $|\arg s| < \pi/2$, $l > n$, $R(s) > 0$, (2μ) is not zero or an integer,

$$\operatorname{Re}[a+\mu+r+\tfrac{5}{4}] > 0,\ \operatorname{Re}\left[a+1-\frac{l}{n}(\alpha_h+\eta_h+c)\right] > l+1,$$

$$h = 1, \ldots, m$$

and the integrals involved are absolutely and uniformly convergent.

As Meijer's G-function is quite a general function a number of particular cases of interest can be obtained from these results after specializing the parameters.

References

Bhise, V. M. (1959). Inversion formula for a generalized Laplace integral. *J. Vikram Univ.*, **3**, No. 3.

Bose, S. K. (1949). A study of the generalized Laplace integral—I. *Bull. Calcutta math. Soc.*, **41**, 9-27.

Mathai, A. M., and Saxena, R. K. (1973). Lecture Notes in Mathematics. Springer-Verlag, Berlin.

Meijer, C. S. (1941a). Eine neue Erweiterung der Laplace Transformation (Erste Mitteilung). *Proc. Nederl. Akad. Wet. Amsterdam*, **44**, 727-37.

———— (1941b). Integralderstellungen fur Whittakersche Funktionen und ihre Produkte (Zweite Mitteilung). *Proc. Nederl. Akad. Wet. Amsterdam*, **44**, 599-605.

Saxena, R. K. (1960). Some theorems on generalized Laplace Transform-I. *Proc. natn. Inst. Sci., India*, **26**, No. 4, 400-13.

Varma, R. S. (1947). A generalization of Laplace transform. *Curr. Sci.*, **16**, 17.

———— (1951). On a generalization of Laplace integral. *Proc. natn. Acad. Sci., India*, **20**, 209-16.