Indian J. pure appl. Math., **10**(7) : 886-889, July 1979

FIXED POINTS FOR A PAIR OF MAPPINGS

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(*Received 2 September* 1978)

Fixed point theorems for a pair of continuous self-mappings of a metric space have been presented in this note.

Recently Dass and Gupta (1975) have proved the following theorem:

Theorem A — Let f be a mapping of a metric space (X, d) into itself such that,

(i) $$d(f(x), f(y)) \leqslant \frac{\alpha d(y, f(y))\,[1 + d(x, f(x))]}{1 + d(x, y)} + \beta d(x, y),$$

$$\forall\ x, y \in X, \alpha, \beta > 0, \alpha + \beta < 1 \text{ and}$$

(ii) for some $x_0 \in X$, the sequence of iterates $\{f^n(x_0)\}$ has a subsequence $\{f^{n_k}(x_0)\}$ with $\xi = \lim\limits_{n \to \infty} f^{n_k}(x_0)$; $\xi \in X$.

Then ξ is a unique fixed point of f.

In this note we extend the above theorem for a pair of mappings in the following way:

Theorem 1 — Let T_1 and T_2 be two continuous self mappings of a metric space (X, d) such that,

$$d(T_1 x, T_2 y) < \frac{\alpha d(y, T_2 y)\,[1 + d(x, T_1 x)]}{1 + d(x, y)} + \beta d(x, y),$$

$$\forall\ x, y \in X, \alpha + \beta = 1, \alpha > 0, \beta > 0.$$

If for some $x_0 \in X$, the sequence $\{x_n\}$ of elements x_n where $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$ ... has a subsequence $\{x_{n_k}\}$ converging to a point $\xi \in X$, then ξ is a unique fixed point of T_1 and T_2.

PROOF : Here $d(x_{2n+1}, x_{2n+2}) = d(T_1 x_{2n}, T_2 x_{2n+1})$

$$< \frac{\alpha d(x_{2n+1}, T_2 x_{2n+1})\,[1 + d(x_{2n}, T_1 x_{2n})]}{1 + d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1})$$

$$= \frac{\alpha d(x_{2n+1}, x_{2n+2})\,[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1})$$

$$\therefore \quad d(x_{2n+1}, x_{2n+2}) < \frac{\beta}{1-\alpha} d(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$$

$$\therefore \quad d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}). \quad \text{...(1)}$$

Again,

$$d(x_{2n}, x_{2n+1}) = d(T_1x_{2n}, T_2x_{2n-1})$$

$$< \frac{\alpha d(x_{2n-1}, T_2x_{2n-1}) [1 + d(x_{2n}, T_1x_{2n})]}{1 + d(x_{2n}, x_{2n-1})} + \beta d(x_{2n}, x_{2n-1})$$

$$\therefore \quad d(x_{2n}, x_{2n+1}) < \frac{\alpha d(x_{2n-1}, x_{2n})}{1 + d(x_{2n}, x_{2n-1})} + \frac{\alpha d(x_{2n-1},x_{2n})\, d(x_{2n}, x_{2n+1})}{1 + d(x_{2n}, x_{2n-1})} + \beta d(x_{2n}, x_{2n-1})$$

or,

$$d(x_{2n}, x_{2n+1}) \left[1 - \frac{\alpha d(x_{2n-1}, x_{2n})}{1 + d(x_{2n}, x_{2n-1})}\right] < d(x_{2n}, x_{2n-1}) \left[\beta + \frac{\alpha}{1 + d(x_{2n}, x_{2n-1})}\right]$$

$$\therefore \quad d(x_{2n}, x_{2n+1}) < \frac{P}{Q} d(x_{2n}, x_{2n-1}) \quad \text{...(2)}$$

where

$$P = \beta + \frac{\alpha}{1 + d(x_{2n}, x_{2n-1})}, \; Q = 1 - \frac{\alpha d(x_{2n-1}, x_{2n})}{1 + d(x_{2n}, x_{2n-1})}.$$

If possible let $P > Q$.

$$\therefore \quad \beta + \frac{\alpha}{1 + d(x_{2n}, x_{2n-1})} > 1 - \frac{\alpha d(x_{2n-1}, x_{2n})}{1 + d(x_{2n}, x_{2n-1})}$$

or,

$$1 + \beta d(x_{2n}, x_{2n-1}) > 1 + d(x_{2n}, x_{2n-1}) (1 - \alpha)$$

$$\therefore \quad d(x_{2n}, x_{2n-1}) < \frac{\beta}{1-\alpha} d(x_{2n}, x_{2n-1}) = d(x_{2n}, x_{2n-1}),$$

which is a contradiction.

$\therefore \; P \leqslant Q.$

Hence from relations (1) and (2)

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n-1}).$$

Continuing this process we get,

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) < d(x_{2n}, x_{2n-1}) < \ldots < d(x_0, x_1).$$

So we have a monotone sequence of positive real numbers which must converge to a real number λ (say).

Since $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in X which converges to some $\xi \in X$, we may write, $\lim x_{2n_k} = \xi$.

If possible let $\xi \neq T_1\xi$.

Now $\quad d(\xi, T_1\xi) = d(\lim x_{2n_k}, T_1 \lim x_{2nk})$

$$= \lim d(x_{2n_k}, T_1 x_{2n_k})$$

$$= \lim d(x_{2n_k}, x_{2n_k+1})$$

$$= \lim d(x_{2n_k+1}, x_{2(n_k+1)}) = d(T_1\xi, T_2T_1\xi).$$

But $$d(T_1\xi, T_2T_1\xi) < \frac{\alpha d(T_1\xi, T_2T_1\xi)\,[1 + d(\xi, T_1\xi)]}{1 + d(\xi, T_1\xi)} + \beta d(\xi, T_1\xi)$$

$$= \alpha d(T_1\xi, T_2T_1\xi) + \beta d(\xi, T_1\xi)$$

$\therefore$ $$d(T_1\xi, T_2T_1\xi) < \frac{\beta}{1-\alpha} d(\xi, T_1\xi) = d(\xi, T_1\xi)$$

$\therefore$ $\quad d(\xi, T_1\xi) = d(T_1\xi, T_2T_1\xi) < d(\xi, T_1\xi)$, a contradiction

$\therefore$ $\quad \xi = T_1\xi$, i.e. ξ is a fixed point of T_1.

Suppose $\quad \xi \neq T_2\xi$

Again $\quad d(\xi, T_2\xi) = d(T_1\xi, T_2T_1\xi) < \alpha d(\xi, T_2\xi)$, a contradiction.

Hence ξ is a fixed point of T_2 also.

$\therefore$ ξ is a common fixed point of T_1 and T_2. Now to prove the uniqueness of ξ, if possible let $\eta (\neq \xi)$ be another fixed point of T_1 and T_2.

Then $$d(\xi, \eta) = d(T_1\xi, T_2\eta) < \frac{\alpha d(\eta, T_2\eta)\,[1 + d(\xi, T_1\xi)]}{1 + d(\xi, \eta)} + \beta d(\xi, \eta)$$

$$= \beta d(\xi, \eta) < d(\xi, \eta)$$

$\therefore$ ξ is a unique common fixed point of T_1 and T_2.

This completes the proof of the theorem.

Theorem 2 — Let T_1 and T_2 be two continuous self-mappings of a metric space (X, d) such that,

$$d(T_1^p x, T_2^q y) < \frac{\alpha d(y, T_2^q y)\,[1 + d(x, T_1^p x)]}{1 + d(x, y)} + \beta d(x, y),$$

$\forall\, x, y \in X$ and $\alpha + \beta = 1$, $p, q > 0$ are integers.

If for some $x_0 \in X$, the sequence $\{x_n\}$ of elements x_n where $x_{2n+1} = T_1^p x_{2n}$, $x_{2(n+1)} = T_2^q x_{2n+1} \ldots$ has a subsequence $\{x_{n_k}\}$ converging to a point $\xi \in X$, then ξ is a unique fixed point of T_1 and T_2.

PROOF : By Theorem 1, T_1^p and T_2^q have a unique fixed point $\xi \in X$.

Now $T_1^p \xi = \xi$ and $T_2^q \xi = \xi$.

Hence $T_1^p (T_1\xi) = T_1(T_1^p \xi) = T_1\xi$

i.e. $T_1\xi$ is a fixed point of T_1^p. But ξ is a unique fixed point of T_1^p. $\therefore$ $T_1\xi = \xi$.

Again $T_2^q (T_2\xi) = T_2(T_2^q \xi) = T_2\xi$

i.e. $T_2\xi$ is a fixed point of T_2^q. Since ξ is a unique fixed point of T_2^q, $\xi = T_2\xi$.

$\therefore$ ξ is a fixed point of T_1 and T_2

To prove the uniqueness, let $\xi \neq \eta$ be another fixed point of T_1 and T_2;

$$d(\xi, \eta) = d(T_1^p \xi, T_2^q \eta) < \frac{\alpha d(\eta, T_2^q \eta)\,[1 + d(\xi, T_1^p \xi)]}{1 + d(\xi, T_1^p \xi)} + \beta d(\xi, \eta)$$

$$= \beta d(\xi, \eta) < d(\xi, \eta),$$

which is a contradiction.

Hence ξ is a unique fixed point of T_1 and T_2.

Hence the theorem.

ACKNOWLEDGEMENT

The author is thankful to Dr B. K. Ray for his kind help and guidance in preparing this paper.

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